Math Methods I
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Fourier Series. Fourier Transform

Fourier Series.

Recall that a function differentiable any number of times at $x = a$ can be represented as a power series

$$\sum_{n=0}^{\infty} a_n(x-a)^n \quad \text{where the coefficients are given by} \quad a_n = \frac{f^{(n)}(a)}{n!}$$

Thus, the function can be approximated by a polynomial. Since this formula involves the n-th derivative, the function f should be differentiable n-times at a. So, just functions that are differentiable any number of times have representation as a power series. This condition is pretty restrictive because any discontinuous function is not differentiable. The functions frequently considered in signal processing, electrical circuits and other applications are discontinuous. Thus, there is a need for a different kind of series approximation of a given function.

The type of series that can represent a a much larger class of functions is called **Fourier Series**. These series have the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{2\pi nx}{T} + b_n \sin\frac{2\pi nx}{T})$$

The coefficients a_n and b_n are called **Fourier coefficients**.

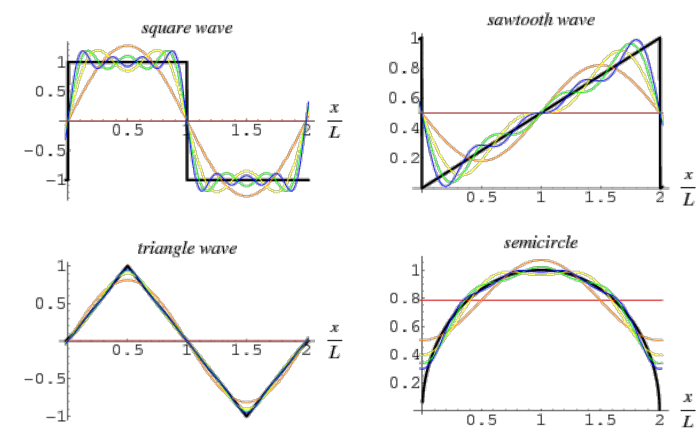
Note that this series represents a **periodic function** with period T.

To represent function $f(x)$ in this way, the function has to be (1) periodic with just a finite number of maxima and minima within one period and just a finite number of discontinuities, (2) the integral over one period of $|f(x)|$ must converge. If these conditions are satisfied and one period of $f(x)$ is given on an interval $(x_0, x_0 + T)$, the Fourier coefficients a_n and b_n can be computed using the formulas

$$a_n = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\cos\frac{2n\pi x}{T}\,dx \qquad\qquad b_n = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\sin\frac{2n\pi x}{T}\,dx$$

When studying phenomena that are periodic in time, the term $\frac{2\pi}{T}$ in the above formula is usually replaced by ω and t is used to denote the independent variable. Thus,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$



If the interval (x_0, x_0+T) is of the form $(-L, L)$ (thus $T = 2L$), then the coefficients a_n and b_n can be computed as follows.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad\qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L})$$

As a special case, if a function is periodic on $[-\pi, \pi]$, these formulas become:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

Obtaining the formulas for coefficients. If $f(x)$ has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{2\pi nx}{T} + b_n\sin\frac{2\pi nx}{T})$$

one can prove the formulas for Fourier series coefficients a_n by multiplying this formula by $\cos\frac{2\pi nx}{T}$ and integrating over one period (say that it is $(\frac{-T}{2}, \frac{T}{2})$).

$$\int_{-T/2}^{T/2} f(x)\cos\frac{2\pi nx}{T}dx = \int_{-T/2}^{T/2}\frac{a_0}{2}\cos\frac{2\pi nx}{T}dx +$$
$$\sum_{m=1}^{\infty}\left(\int_{-T/2}^{T/2} a_m\cos\frac{2\pi mx}{T}\cos\frac{2\pi nx}{T}dx + \int_{-T/2}^{T/2} b_m\sin\frac{2\pi mx}{T}\cos\frac{2\pi nx}{T}dx\right)$$

All the integrals with $m \neq n$ are zero and the integrals with both sin and cosine functions are zero as well. Thus,

$$\int_{-T/2}^{T/2} f(x)\cos\frac{2\pi nx}{T}dx = \int_{-T/2}^{T/2} a_n\cos\frac{2\pi nx}{T}\cos\frac{2\pi nx}{T}dx = a_nT \Rightarrow a_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\cos\frac{2\pi nx}{T}dx$$

if $n > 0$. If $n = 0$, we have that $Ta_0 = \int_{-T/2}^{T/2} f(x)dx$ and so $a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)dx$.

The formula for b_n is proved similarly, multiplying by $\sin\frac{2\pi nx}{T}$ instead of $\cos\frac{2\pi nx}{T}$.

Symmetry considerations. Note that if $f(x)$ is even (that is $f(-x) = f(x)$), then $b_n = 0$. Since $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\, dx = \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\, dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\, dx$. Using the substitution $u = -x$ for the first integral we obtain that it is equal to $\frac{-1}{L}\int_{L}^{0} f(-u)\sin\frac{-n\pi u}{L}\, du$. Using that $f(-x) = f(x)$, that $\sin\frac{-n\pi u}{L} = -\sin\frac{n\pi u}{L}$, and that $-\int_L^0 = \int_0^L$, we obtain $\frac{1}{L}\int_0^L f(u)(-\sin\frac{n\pi u}{L})\, du = \frac{-1}{L}\int_0^L f(u)\sin\frac{n\pi u}{L}\, du$. Note that this is exactly the negative of the second integral. Thus, the first and the second integral cancel and we obtain that $b_n = 0$.

Similarly, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx = \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\, dx + \frac{1}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\, dx$. Using the substitution $u = -x$ for the first integral we obtain that it is equal to $\frac{-1}{L}\int_{L}^{0} f(-u)\cos\frac{-n\pi u}{L}\, du$. Using that $f(-x) = f(x)$, that $\cos\frac{-n\pi u}{L} = \cos\frac{n\pi u}{L}$, and that $-\int_L^0 = \int_0^L$, we obtain $\frac{1}{L}\int_0^L f(u)\cos\frac{n\pi u}{L}$. Note that this is equal to the second integral. Thus, the two integrals can be combined and so

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx \quad \text{and} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$

If $f(x)$ is odd, using analogous arguments, we obtain that $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

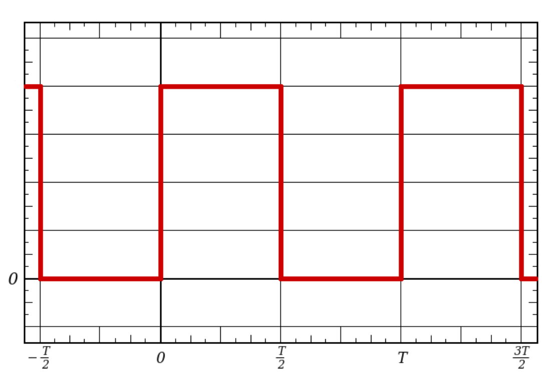
These last two power series are called **Fourier cosine expansion** and **Fourier sine expansion** respectively.

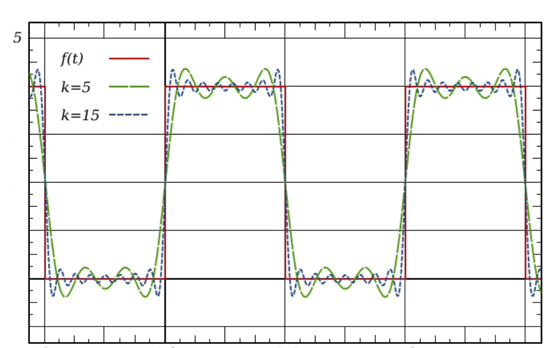
Example 1. The input to an electrical circuit that switches between a high and a low state with time period 2π can be represented by the **boxcar function**.

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ -1 & -\pi \le x < 0 \end{cases}$$

The periodic expansion of this function is called the **square wave function**.

More generally, the input to an electrical circuit that switches from a high to a low state with time period T can be represented by the **general square wave function** with the following formula on the basic period. $f(x) = \begin{cases} 1 & 0 \le x < \frac{T}{2} \\ -1 & \frac{-T}{2} \le x < 0 \end{cases}$





Find the Fourier series of the square wave and the general square wave.

Solutions. Graph the square wave function and note it is odd. Thus, the coefficients of the cosine terms will be zero. Since $L = \pi$ ($T = 2\pi$), the coefficients of the sine terms can be computed as $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{-2}{n\pi}\cos nx|_0^{\pi} = \frac{-2}{n\pi}((-1)^n - 1)$. Note that $(-1)^n - 1 = 1 - 1 = 0$ if n is even (say $n = 2k$) and $(-1)^n - 1 = -1 - 1 = -2$ if n is odd (say $n = 2k+1$). Thus, $b_{2k} = 0$ and $b_{2k+1} = \frac{-2}{n\pi}(-2) = \frac{4}{(2k+1)\pi}$. Hence,

$f(x) = \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{1}{n}\sin nx = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1} = \frac{4}{\pi}(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots)$.

For the general square wave, analogously to this previous consideration you obtain that $a_n = 0$, $b_{2k} = 0$ and $b_{2k+1} = \frac{4}{(2k+1)\pi}\sin\frac{2(2k+1)\pi x}{T}$. Thus, $f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\frac{2(2k+1)\pi x}{T}$.

Even and odd extensions. If an arbitrary function $f(x)$, not necessarily even or odd, is defined on the interval $(0, L)$, we can extend it to an even function

$$g(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \end{cases}$$

and consider its Fourier cosine expansion:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{and} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$

Similarly, we can extend $f(x)$ to an odd function

$$h(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \end{cases}$$

and consider its Fourier sine expansion:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

Useful formulas. To simplify the answers, sometimes the following identities may be useful

$\sin n\pi = 0$	$\cos n\pi = (-1)^n$

In particular, $\cos 2n\pi = 1$ and $\cos(2n+1)\pi = -1$.

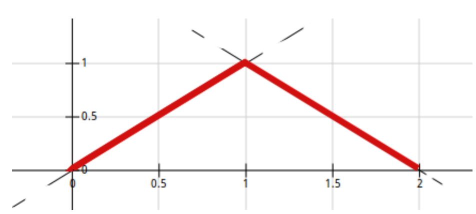
If $n = 2k+1$ is an odd number,

$\sin\frac{n\pi}{2} = \sin\frac{(2k+1)\pi}{2} = (-1)^k$	$\cos\frac{n\pi}{2} = \cos\frac{(2k+1)\pi}{2} = 0$

Example 2.

Find the Fourier cosine expansion of

$$f(x) = \begin{cases} x & 0 < x \le 1 \\ 2 - x & 1 < x < 2 \end{cases}$$



Solutions. First extend the function symmetrically with respect to y-axis so that it is defined on basic period [-2,2] and that it is *even.* Thus $T=4$ and $L=2$. The coefficients b_n are zero in this case and the coefficients a_n can be computed as follows. $a_n=\int_0^2 f(x)\cos\frac{n\pi x}{2}dx=\int_0^1 x\cos\frac{n\pi x}{2}dx+\int_1^2(2-x)\cos\frac{n\pi x}{2}dx$. Using integration by parts with $u=x$, $v=\frac{2}{n\pi}\sin\frac{n\pi x}{2}$ for the first and $u=2-x$ and same v for the second, you obtain $a_n=\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2}+\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)|_0^1+\left(\frac{2(2-x)}{n\pi}\sin\frac{n\pi x}{2}-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)|_1^2=$ $\underbrace{\frac{2}{n\pi}\sin\frac{n\pi}{2}}+\underline{\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}}-\frac{4}{n^2\pi^2}-\frac{4}{n^2\pi^2}\cos n\pi-\underbrace{\frac{2}{n\pi}\sin\frac{n\pi}{2}}+\underline{\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}}=\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{4}{n^2\pi^2}-\frac{4}{n^2\pi^2}\cos n\pi=$ $\frac{4}{n^2\pi^2}(2\cos\frac{n\pi}{2}-1-\cos n\pi)$.

If $n=2k+1$ is odd, $a_n=\frac{4}{(2k+1)^2\pi^2}(0-1+1)=0$. If $n=2k$ is even, $a_n=\frac{4}{(2k)^2\pi^2}(2(-1)^k-1-1)$. Because of the part with $(-1)^k$, we can distinguish two more cases depending on whether k is even or odd. Thus, if $k=2l$ is even, $a_n=\frac{4}{(4l)^2\pi^2}(2-1-1)=0$. If $k=2l+1$ is odd, $a_n=\frac{4}{(2(2l+1))^2\pi^2}(2(-1)-1-1)=\frac{-16}{(4l+2)^2\pi^2}=\frac{-4}{(2l+1)^2\pi^2}$.

If $n=0$, $a_0=\int_0^2 f(x)dx=\int_0^1 xdx+\int_1^2(2-x)dx=\frac{1}{2}+\frac{1}{2}=1$.

So, $f(x)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\cos\frac{(4l+2)\pi x}{2}=\frac{1}{2}-\frac{4}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\cos(2l+1)\pi x$.

Example 3. Consider the function $f(x)=x^2$ for $0\le x\le 2$.

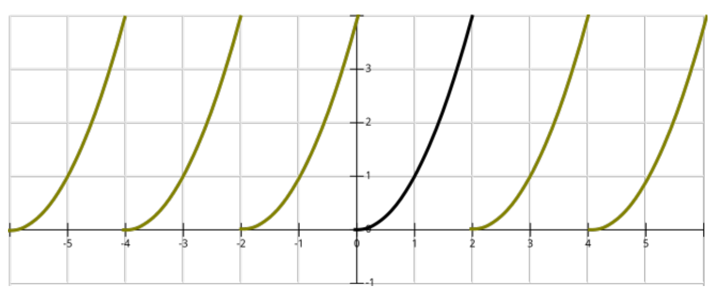
(a) Sketch the graphs of the following (1) the periodic extension of $f(x)$, (2) the even periodic extension of $f(x)$, (3) the odd periodic extension of $f(x)$ and write the integrals computing the coefficients of the corresponding Fourier series in all three cases.

(b) Find the Fourier cosine expansion for $f(x)$.

Solutions. (a) The periodic extension of $f(x)$ is neither even nor odd. You can obtain the graph of it by replicating $f(x)$ on intervals $\ldots[-4,-2],[-2,0],[0,2],[2,4],[4,6]\ldots$ of length $T=2$. The coefficients of the corresponding Fourier series can be calculated by

$$a_n=\int_0^2 x^2\cos n\pi x\;dx$$

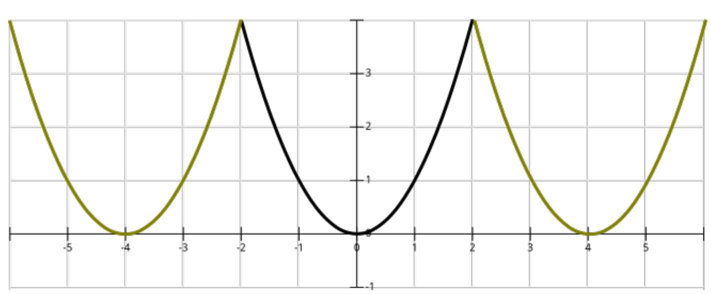
$$b_n=\int_0^2 x^2\sin n\pi x\;dx.$$



The even extension of $f(x)$ is obtained by extending $f(x)$ from [0, 2] to $[-2,2]$ by defining $f(x)=(-x)^2=x^2$ on $[-2,0]$. Thus, the result is the function x^2 defined on interval $[-2,2]$.

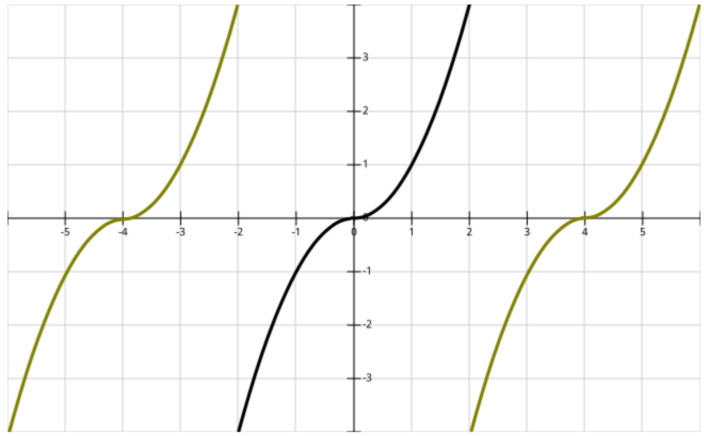
Then, replicate this function on intervals $\ldots[-6,-2],[-2,2],[2,6],[6,10],\ldots$ of length $T=4$. Thus, $T=4$ and $L=2$. The coefficients of the corresponding cosine Fourier series can be calculated by

$$a_n=\int_0^2 x^2\cos\frac{n\pi x}{2}dx \qquad b_n=0.$$



The odd extension of $f(x)$ is obtained by extending $f(x)$ from $[0, 2]$ to $[-2, 2]$ by defining $f(x) = -(x)^2 = -x^2$ on $[-2, 0]$. Thus, the result is the function

$$\begin{cases} x^2 & 0 \leq x \leq 2 \\ -x^2 & -2 \leq x < 0 \end{cases}$$



defined on interval $[-2, 2]$. Replicate this function on intervals $\ldots [-6, -2], [-2, 2], [2, 6], [6, 10], \ldots$ of length $T = 4$. Thus, $T = 4$ and $L = 2$. The coefficients of the corresponding since Fourier series can be calculated by

$$b_n = \int_0^2 x^2 \sin \frac{n\pi x}{2}\, dx \qquad a_n = 0.$$

(b) By part (a), the coefficients a_n can be calculated by $a_n = \int_0^2 x^2 \cos \frac{n\pi x}{2} dx$. Using integration by parts with $u = x^2$ and $v = \int \cos \frac{n\pi x}{2} dx = \frac{2}{n\pi} \sin \frac{n\pi x}{2}$, we obtain that $a_n = \frac{2}{n\pi} x^2 \sin \frac{n\pi x}{2} |_0^2 - \frac{4}{n\pi} \int_0^2 x \sin \frac{n\pi x}{2} dx$. The first term is zero and the second term requires another integration by parts, this time with $u = x$ and $v = \int \sin \frac{n\pi x}{2} dx = \frac{-2}{n\pi} \cos \frac{n\pi x}{2}$. Thus $a_n = \frac{8}{n^2\pi^2} x \cos \frac{n\pi x}{2} |_0^2 - \frac{16}{n^3\pi^3} \sin \frac{n\pi x}{2} |_0^2$. The first term is zero in the lower bound and the second term is zero at both bounds. Thus $a_n = \frac{16}{n^2\pi^2} \cos n\pi = \frac{16(-1)^n}{n^2\pi^2}$. Note that this formula works just for $n > 0$ so a_0 has to be computed separately by $a_0 = \int_0^2 x^2 dx = \frac{x^3}{3} |_0^2 = \frac{8}{3}$.

Thus, the Fourier series is $x^2 = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2}$.

Complex Fourier Series. The complex form of Fourier series is the following:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2n\pi i x}{T}} = \sum_{n=-\infty}^{\infty} c_n \left(\cos \frac{2n\pi x}{T} + i \sin \frac{2n\pi x}{T} \right) \text{ where } c_n = \frac{1}{T} \int_{x_0}^{x_0+T} f(x) e^{\frac{-2n\pi i x}{T}}\, dx.$$

If $f(x)$ is a real function, the coefficients c_n satisfy the relations $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n)$ for $n > 0$. Thus, $c_{-n} = \overline{c_n}$ for all $n > 0$. In addition,

$$a_n = c_n + c_{-n} \text{ and } b_n = i(c_n - c_{-n}) \text{ for } n > 0 \text{ and } a_0 = 2c_0.$$

These coefficients c_n are further associated to $f(x)$ by **Parseval's Theorem**. This theorem is related to conservation law and states that

$$\frac{1}{T} \int_{x_0}^{x_0+T} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Note that the integral on the left side computes the average value of the moduli squared of $f(x)$ over one period and the right side is the sum of the moduli squared of the complex coefficients. The proof of this theorem can be your project topic.

Symmetry Considerations. If $f(x)$ is either even or odd function defined on interval $(-L, L)$, the value of $|f(x)|^2$ on $(-L, 0)$ is the same as the value of $|f(x)|^2$ on $(0, L)$. Thus,

$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \frac{1}{L}\int_0^L |f(x)|^2dx$. In this case, Parseval's Theorem has the following form.

$$\frac{1}{L}\int_0^L |f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

Example 4. Using the Fourier series for $f(x) = x^2$ for $0 < x < 2$ from Example 3 and Parseval's Theorem, find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

Solutions. Recall that $T = 4$, $L = 2$ and that the function is even. In Example 3 we have find that $a_n = \frac{16(-1)^n}{n^2\pi^2}$ for $n > 0$, $a_0 = \frac{8}{3}$. Thus, Parseval's Theorem applied to this function results in the following

$$\frac{1}{2}\int_0^2 x^4dx = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2) = \frac{16}{9}+\frac{16^2}{2\pi^4}\sum_{n=1}^{\infty}\left(\frac{1}{n^4}+0\right).$$

Note that integral on the left side is $\frac{1}{2}\int_0^2 x^4dx = \frac{16}{5}$. Dividing the equation above by 16 produces

$$\frac{1}{5} = \frac{1}{9}+\frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \quad\Rightarrow\quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Practice Problems.

1. Use the Fourier series you obtain in Example 1 to find the sum of series
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

2. Note that the boxcar function from Example 1 represents the odd extension of the function $f(x) = 1$ for $0 \le x < \pi$. Consider the even extension of this function and find its Fourier cosine expansion.

3. Find the Fourier sine expansion of $f(x) = \begin{cases} x & 0 < x \le 1 \\ 2-x & 1 < x < 2 \end{cases}$ Use it to find the sum of series
$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

4. The voltage in an electronic oscillator is represented as a sawtooth function $f(t) = t$ for $0 \le t \le 1$ that keeps repeating with the period of 1. Sketch this function and represent it using a complex Fourier series. Then use this Fourier series and Parseval's Theorem to find the sum of the series
$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

5. Find the Fourier series of $f(x) = \begin{cases} 1-x & 0 \le x < 1 \\ 1+x & -1 < x < 0 \end{cases}$

Solutions.

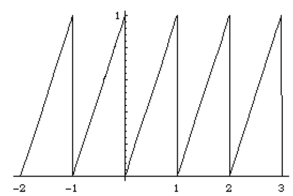
1. In Example 1, we obtain that $f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$ where $f(x)$ is the square wave function. Choosing $x = \frac{\pi}{2}$, we have that $1 = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\frac{\pi}{2}}{2n+1} = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.

2. The even extension of $f(x)$ is $f(x) = 1$ for $-\pi < x < \pi$. The periodic extension of this function is constant function equal to 1 for every x value. Note that this is already in the form of a Fourier series with $b_n = 0$ for every n, $a_n = 0$ for $n > 1$ and $a_0 = 2$. Computing the Fourier coefficients would give you the same answer: $a_n = \frac{2}{\pi}\int_0^{\pi}\cos nx\ dx = 0$ if $n > 1$ and $a_0 = \frac{2}{\pi}\int_0^{\pi} dx = 2$. Thus, $f(x) = \frac{2}{2} + \sum_{n=1}^{\infty} 0 = 1$. This answer should not be surprising since this function is already in the form of a Fourier series $1 = 1 + \sum_{n=0}^{\infty}(0\cos nx + 0\sin nx)$.

3. Extend the function symmetrically about the origin so that it is odd. Thus $T = 4$ and $L = 2$. Since this extension is odd, $a_n = 0$. Compute b_n as $b_n = \int_0^2 f(x)\sin\frac{n\pi x}{2}dx = \int_0^1 x\sin\frac{n\pi x}{2}dx + \int_1^2(2-x)\sin\frac{n\pi x}{2}dx = \left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)|_0^1 + \left(\frac{-2(2-x)}{n\pi}\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)|_1^2 = \underline{\frac{-2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \underline{\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$. This is 0 if n is even. If $n = 2k+1$, this is $\frac{8(-1)^k}{(2k+1)^2\pi^2}$. So, $f(x) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)\pi x}{2}$.

   To find the sum of series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, note that when $x = 1$ the function $f(1)$ is equal to 1 and its Fourier sine expansion is equal to $\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi}{2} = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}(-1)^n = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$ so $\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 1 \Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

4. $T = 1$, $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2n\pi it}$ and $c_n = \int_0^1 te^{-2n\pi it} = \frac{t}{-2n\pi i}e^{-2n\pi it}|_0^1 + \frac{1}{4n^2\pi^2}e^{-2n\pi it}|_0^1 = \frac{1}{-2n\pi i}e^{-2n\pi i} + \frac{1}{4n^2\pi^2}e^{-2n\pi i} - \frac{1}{4n^2\pi^2}$.



   Note that $e^{-2n\pi i} = \cos(-2n\pi) + i\sin(-2n\pi) = 1$. Thus $c_n = \frac{1}{-2n\pi i} + 0 = \frac{i}{2n\pi}$. Note that $c_{-n} = \frac{-i}{2n\pi} = \overline{c_n}$. $c_0 = \int_0^1 t\ dt = \frac{1}{2}$. This gives us $f(t) = \frac{1}{2} + \sum_{n=-\infty, n\neq 0}^{\infty}\frac{i}{2n\pi}e^{2n\pi it}$. Note also that $a_0 = 2c_0 = 1$, $a_n = 0$ for $n > 0$ and $b_n = \frac{-1}{n\pi}$.

   Parseval's Theorem gives us that $\int_0^1 x^2 dx = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} \Rightarrow \frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

5. Graph the function and note it is even. Thus, $b_n = 0$. Since $T = 2$ and $L = 1$ $a_n = 2\int_0^1(1-x)\cos(n\pi x)\ dx$. Using the integration by parts with $u = 1-x$ and $v = \int\cos(n\pi x)dx = \frac{1}{n\pi}\sin(n\pi x)$, obtain that $a_n = \frac{2}{n\pi}(1-x)\sin(n\pi x)|_0^1 + \frac{2}{n\pi}\int_0^1\sin(n\pi x)\ dx = 0 - \frac{2}{n^2\pi^2}\cos(n\pi x)|_0^1 = -\frac{2}{n^2\pi^2}((-1)^n - 1)$. This last expression is 0 if n is even and equal to $\frac{2}{n^2\pi^2} = \frac{2}{(2k+1)^2\pi^2}$ if $n = 2k+1$ is odd. Note that the formula $-\frac{2}{n^2\pi^2}((-1)^n - 1)$ does not compute a_0 because of the n in the denominator so calculate a_0 from the formula $a_0 = 2\int_0^1(1-x)\ dx = x - \frac{x^2}{2}|_0^1 = \frac{1}{2}$. This gives you the Fourier series expansion $f(x) = \frac{1}{4} + \sum_{k=0}^{\infty}\frac{4}{(2k+1)^2\pi^2}\cos(2k+1)\pi x$.

Fourier Transformation

The Fourier transform is an integral operator meaning that it is defined via an integral and that it maps one function to the other. If you took a differential equations course, you may recall that the Laplace transform is another integral operator you may have encountered.

The Fourier transform represents a generalization of the Fourier series. Recall that the Fourier series is $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2n\pi it}{T}}$. The sequence c_n can be regarded as a function of n and is called **Fourier spectrum of** $f(t)$. We can think of $c(n)$ being another representation of $f(t)$, meaning that $f(t)$ and $c(n)$ are different representations of the same object. Indeed: given $f(t)$ the coefficients $c(n)$ can be computed and, conversely, given $c(n)$, the Fourier series with coefficients $c(n)$ defines a function $f(t)$. We can plot $c(n)$ as a function of n (and get a set of infinitely many equally spaced points). In this case we think of c as a function of n, the **wave number**.

We can also think of c as a function of $\omega = \frac{2n\pi}{T}$, the **frequency**. If T is large, then ω is small, so for large T, we can think of $c(\omega)$ being a continuous function. Also, two consecutive n values are length 1 apart so $dn = 1$. Thus, $d\omega = \frac{2\pi}{T} dn \Rightarrow d\omega = \frac{2\pi}{T}$ and $\frac{T d\omega}{2\pi} = 1$. Thus we have

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{\frac{-2n\pi it}{T}} \, dt \quad \Rightarrow \quad Tc(\omega) = \int_{-T/2}^{T/2} f(t) e^{-i\omega t} dt \quad \text{and so}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2n\pi it}{T}} = \sum_{T\omega/(2\pi)=-\infty}^{\infty} \frac{T d\omega}{2\pi} c(\omega) e^{\omega it} = \frac{1}{2\pi} \sum_{T\omega/(2\pi)=-\infty}^{\infty} Tc(\omega) e^{\omega it} d\omega.$$

When we let $T \to \infty$, the above expression become

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right) e^{\omega it} d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right) e^{\omega it} d\omega$$

We denote the expression in parenthesis by $F(\omega)$ and so get the final formulas:

$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$	$F(\omega)$ is the **Fourier transform** of $f(t)$.
$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$	$f(t)$ is the **inverse Fourier transform** of $F(\omega)$.

We can still think of $f(t)$ and $F(\omega)$ being the same representations of the same object: the first formula above computes $F(\omega)$ for given $f(t)$ and the second one computes $f(t)$ for given $F(\omega)$. To understand the significance of this, it is helpful to think of $f(t)$ as of a signal which can be measured in time (so $f(t)$ can be obtained) but that needs to be represented as a function of frequency, not time. In this case, Fourier transform produces representation of $f(t)$ as a function $F(\omega)$ of frequency ω.

The Fourier transform is not limited to functions of time and temporal frequencies. It can be used to analyze spatial frequencies. If the independent variable in $f(t)$ stands for space instead of time, x is usually used instead of t. In this case, the independent variable of the inverse transform is denoted by k.

$$t \longleftrightarrow x \qquad \text{and} \qquad \omega \longleftrightarrow k$$

Mathematically, the importance of the Fourier transform lies in the following:

1. If the initial function $f(t)$ has the properties that are not desirable in a particular application (e.g. discontinuous, non smooth), we can consider the function $F(\omega)$ instead which is possible better behaved.

2. Fourier transform represents a function that is not necessarily periodic and that is defined on infinite interval. The only requirement for the Fourier transform to exist is that the integral $\int_{-\infty}^{\infty} |f(t)|dt$ is convergent.

3. Fourier transformation is a generalization of Fourier series.

In physics, on the other hand, Fourier transform is used in many sub-disciplines. One of the most important applications of Fourier transform is in **signal processing**. We have pointed out that the Fourier transform presents the signal $f(t)$, measured and expressed as functions of time, as a function $F(\omega)$ of frequency. The transform $F(\omega)$ is also known as the **frequency spectrum** of the signal.

Moreover, Fourier transform provides information on the amplitude and phase of a source signal at various frequencies. The transform $F(\omega)$ of a signal $f(t)$ can be written in polar coordinates as

$$F = |F|e^{\theta i}$$

The modulus $|F|$ represents the **amplitude** of the signal at respective frequency ω, while θ (given by $\arctan(\text{ Im } F/ \text{ Re } F)$ computes the phase shift at frequency ω. This gives rise to various applications cryptography, acoustics, optics and other areas.

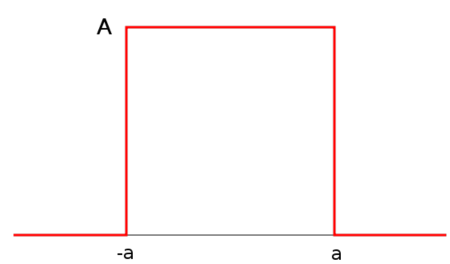
In addition, Fourier transform can be used similarly to Laplace transform: in converts a differential equation into an ordinary (algebraic) equation that is easier to solve. After solving it, the solution of the original differential equation can be obtained by using the inverse Fourier transform.

Example 1. Find the Fourier transform $F_1^1(\omega)$ of the **boxcar function**.

$$f_1^1(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

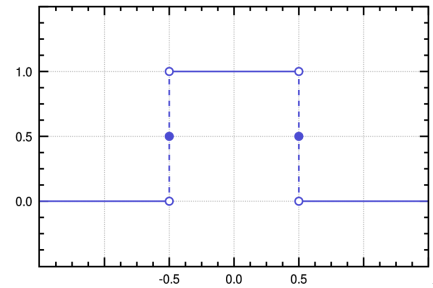
Express your answer as real function.

Generalize your calculations to find the Fourier transform $F_a^A(\omega)$ for the general boxcar function.
$f_a^A(t) = \begin{cases} A & -a < t < a \\ 0 & \text{otherwise} \end{cases}$



The boxcar function is said to be **normalized** if $A = \frac{1}{2a}$ so that the total area under the function is 1. We shall denote the normalized boxcar function that is non-zero on interval $[-a, a]$ by f_a.

Sometimes the values of f_a at $x = a$ and $x = -a$ are defined to be $\frac{1}{4a}$. For example, if $a = \frac{1}{2}$, the following graph represents f_a.



Consider how changes in value of a impact the shape and values of $F_a(\omega)$.

Solution. $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$. Since $f(t) = 0$ for $t < -1$ and $t > 1$, and $f(t) = 1$ for $-1 \leq t \leq 1$, we have that

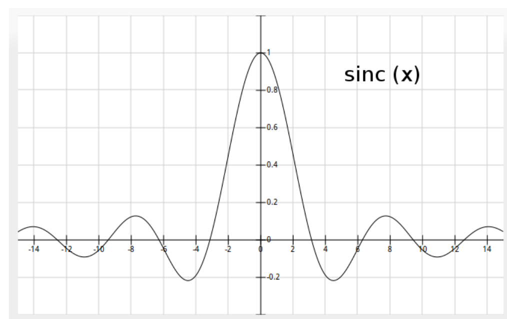
$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega t} dt = \frac{-1}{\sqrt{2\pi} i\omega} e^{-i\omega t}|_{-1}^{1} = \frac{-1}{\sqrt{2\pi} i\omega}(e^{-i\omega} - e^{i\omega})$. Use the Euler's formula to obtain $F(\omega) = \frac{-1}{\sqrt{2\pi} i\omega} (\cos(-\omega) + i\sin(-\omega) - \cos\omega - i\sin\omega)$. Using that $\cos(-\omega) = \cos\omega$ since cosine is even and $\sin(-\omega) = -\sin\omega$ since sine is odd function, the cosine terms cancel and we obtain that

$F(\omega) = \frac{-1}{\sqrt{2\pi} i\omega} (-2i\sin\omega) = \frac{2}{\sqrt{2\pi}\omega} \sin\omega$.

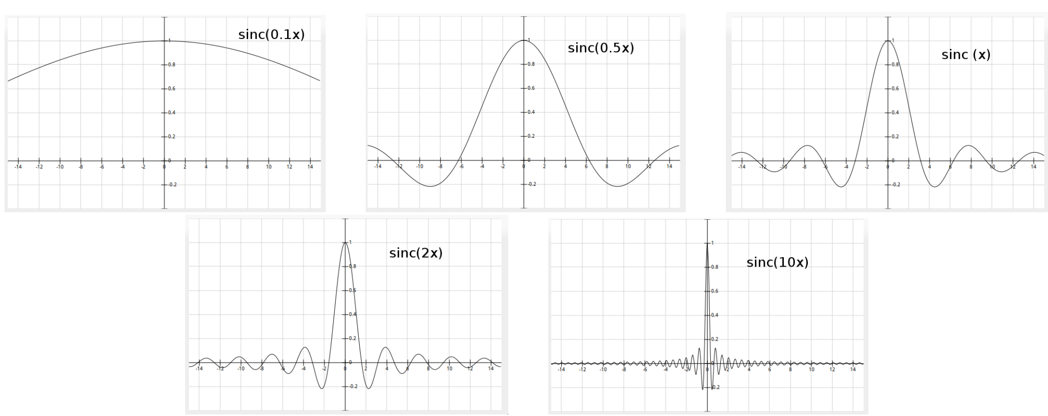
Similarly, if a boxcar function of height A is nonzero on interval $[-a, a]$, the Fourier transform is $F_a^A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} A e^{-i\omega t} dt = \frac{A}{\sqrt{2\pi} i\omega}(e^{-ia\omega} - e^{ia\omega}) = \frac{2A}{\sqrt{2\pi}\omega} \frac{e^{ia\omega} - e^{-ia\omega}}{2i} = \frac{2A}{\sqrt{2\pi}\omega} \sin a\omega$.

Function $\frac{\sin x}{x}$ is known as **sinc function**.

$$\operatorname{sinc}(x) = \frac{\sin x}{x}$$



Sometimes the **normalized sinc function** $\frac{\sin \pi x}{\pi x}$ is used. Note that $\lim_{x\to 0} \operatorname{sinc}(x) = 1$ and $\lim_{x\to\infty} \operatorname{sinc}(x) = 0$.



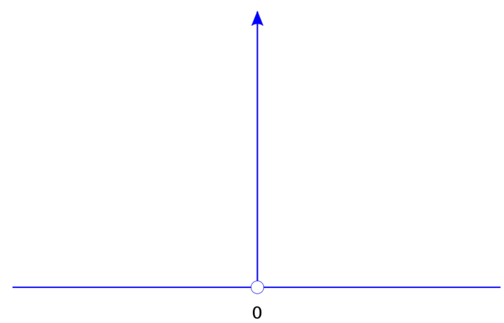
Using the sinc notation, we can represent

the Fourier transform of the boxcar function $f_a^A(t)$ as the sinc function $F_a^A(\omega) = \frac{2aA}{\sqrt{2\pi}} \operatorname{sinc}(a\omega)$.

If the box function is normalized, $A = \frac{1}{2a}$ so $F_a(\omega) = \frac{1}{\sqrt{2\pi}}\text{sinc}(a\omega)$. Since sinc(0)=1, the peak of this function has value $\frac{1}{\sqrt{2\pi}}$.

Let us compare the graphs of f_a and F_a for several different values of a. You can notice that larger values of a correspond to f_a having smaller hight and being more spread out. In this case, F_a has a narrow, sharp peak at $\omega = 0$ and converges to 0 faster. If a is small, f_a has large height and very narrow base. In this case, F_a has very spread out peak around $\omega = 0$ and the convergence to 0 is much slower.

In the limiting cases $a \to 0$ represents a constant function and $a \to \infty$ represents **the Dirac delta function** $\delta(t)$. This function, known also as the **impulse function** is used to represent phenomena of an impulsive nature. For example, voltages that act over a very short period of time. It is defined by



$$\delta(t) = \lim_{a \to 0} f_a(t).$$

Since the area under $f_a(t)$ is 1, the area under $\delta(t)$ is 1 as well. Thus, $\delta(t)$ is characterized by the following properties:

$$(1) \quad \delta(t) = 0 \text{ for all values of } t \neq 0 \qquad (2) \quad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

Since no ordinary function satisfies both of these properties, δ is not a function in the usual sense of the word. It is an example of a generalized function or a distribution. Alternate definition of Dirac delta function can be found on wikipedia.

We have seen that F_a for a small a is very spread out and almost completely flat. So, in the limiting case when $a = 0$, it becomes a constant. Thus, the Fourier transform of $\delta(t)$ can be obtained as limit of $F_a(\omega)$ when $a \to 0$. We have seen that this is a constant function passing $\frac{1}{\sqrt{2\pi}}$. Thus,

> the Fourier transform of the delta function $\delta(t)$ is the constant function $F(\omega) = \frac{1}{\sqrt{2\pi}}$.

Conversely, when $a \to \infty$ $f_a \to \frac{1}{\sqrt{2\pi}}$. The Fourier transform of that is limit of F_a for $a \to \infty$. We have seen that this limit function is zero at all nonzero values of ω. So, we can relate this limit to $\delta(\omega)$. Requiring the area under F_a to be 1 can explain why normalized sinc function is usually considered instead of regular sinc.

Example 2. Find the inverse Fourier transforms of boxcar and Dirac delta functions.

Solutions. The inverse transform of the general boxcar function can be computed by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} Ae^{i\omega t}d\omega = \frac{A}{\sqrt{2\pi}it}(e^{iat} - e^{-iat})$$

To express this answer as a real function, use the Euler's formula and symmetries of sine and cosine functions similarly as in Example 1 to obtain the following.

$f(t) = \frac{A}{\sqrt{2\pi} it} (\cos at + i \sin at - \cos(-at) - i \sin(-at)) = \frac{A}{\sqrt{2\pi} it} (\cos at + i \sin at - \cos at + i \sin at) = \frac{A}{\sqrt{2\pi} it} (2i \sin at) = \frac{2A}{\sqrt{2\pi} t} \sin at = \frac{2aA}{\sqrt{2\pi} t} \frac{\sin at}{a} = \frac{2aA}{\sqrt{2\pi}} \frac{\sin at}{at} = \frac{2aA}{\sqrt{2\pi}} \text{sinc}(at)$. Thus, the inverse transform of the boxcar function is the sinc function. This also illustrates that

> the Fourier transform of the sinc function is the boxcar function.

Taking the limit when $a \to 0$ in the normalized case, we obtain that the inverse Fourier of the delta function is the limit of $\frac{1}{\sqrt{2\pi}} \text{sinc}(at)$ when $a \to 0$ which is the constant function $\frac{1}{\sqrt{2\pi}}$. Thus,

> the Fourier transform of the constant function $f(t) = \frac{1}{\sqrt{2\pi}}$ is the delta function $\delta(\omega)$.

Example 3. Consider the **Gaussian probability function** (a.k.a. the "bell curve")

$$f(t) = Ne^{-at^2}$$

where N and a are constants. N determines the height of the peak and a determines how fast it decreases after reaching the peak. The Fourier transform of $f(t)$ can be found to be

$$F(\omega) = \frac{N}{\sqrt{2a}} e^{-\omega^2/4a}.$$

This is another Gaussian probability function. Examine how changes in a impact the graph of the transform.

Solutions. If a is small, f is flattened. In this case the presence of a in the denominator of the exponent of $F(\omega)$ will cause the F to be sharply peaked and the presence of a in the denominator of $\frac{N}{\sqrt{2a}}$, will cause F to have high peak value.

If a is large, f is sharply peaked and F is flattened and with small peak value.

All the previous examples are related to Heisenberg uncertainty principle and have applications in quantum mechanics. The following table summarize our current conclusions.

Function in time domain	Fourier Transform in frequency domain
boxcar function	sinc function
sinc function	boxcar function
delta function	constant function
constant function	delta function
Gaussian function	Gaussian function

Symmetry considerations.

The table on the right displays the formulas of Fourier and inverse Fourier transform for even and odd functions.

$f(t)$ even	$F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t \, dt$
$F(\omega)$ even	$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega) \cos \omega t \, d\omega$
$f(t)$ odd	$F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t \, dt$
$F(\omega)$ odd	$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega) \sin \omega t \, d\omega$

Fourier sine and cosine functions. Suppose that $f(t)$ is defined just for $t > 0$. We can extend $f(t)$ so that it is even. Then we get the formula for $F(\omega)$ by using the formulas for even function above. $F(\omega)$ is then called **Fourier cosine transformation.**

Similarly, if we extend $f(t)$ so that it is odd, we get the formula for $F(\omega)$ same as for an odd function above. $F(\omega)$ is then called **Fourier sine transformation.**

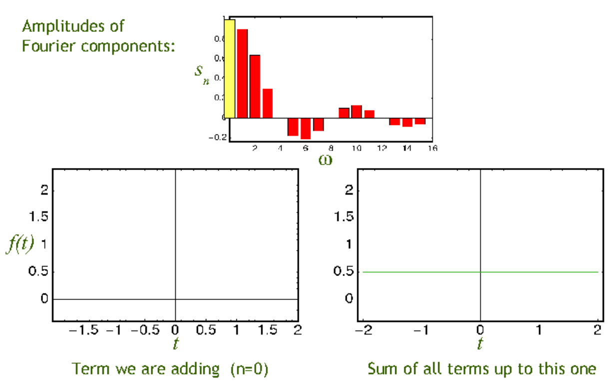
Relation of Fourier transform and Fourier Series.

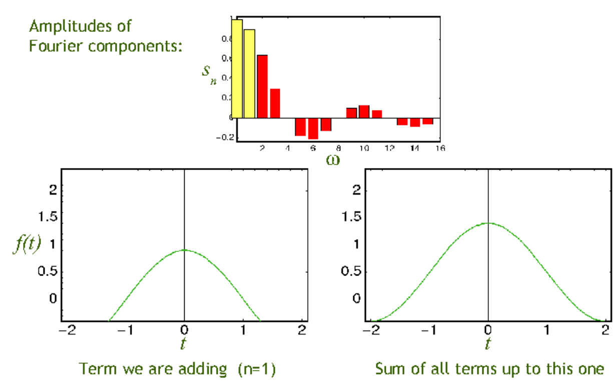
We illustrate this relation in the following example. Consider the Fourier series of a boxcar function $f_a(t)$. Let s_n denote the Fourier coefficient in the complex Fourier series. The value of a determines sampling period T and spacing ω_0 in frequency-domain.

$$w_0 = \frac{2\pi}{T} \text{ and } \omega = \frac{2n\pi}{T} \Rightarrow \omega = n\omega_0.$$

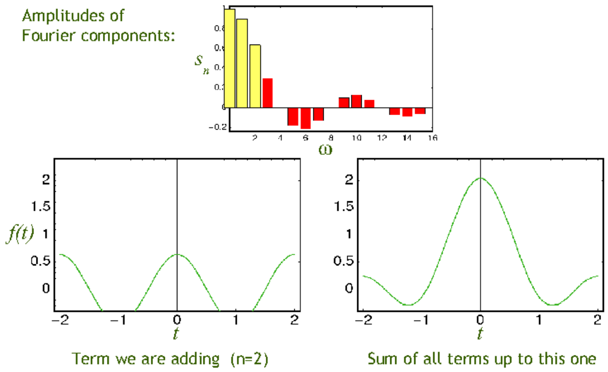
In the following figures [1], we consider the boxcar function with $a = \frac{1}{2}$ (so $T = 1$ and $\omega_0 = 2\pi$). The first graph on each figure displays the Fourier transform, sinc function, in the frequency-domain. The highlighted frequency on the first graph determines the value of n. The second graph displays the harmonic function corresponding to n-th term of the Fourier series of the function $f(t)$ in the time-domain. The third graph is the sum of the first n terms of the Fourier series of $f(t)$.

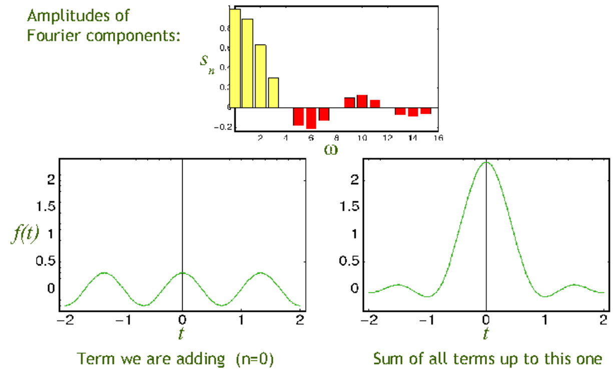
Note that as $n \to \infty$ the sum of Fourier series terms converge to the boxcar function.

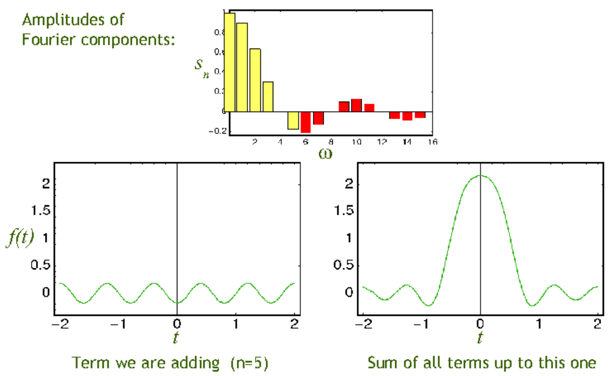


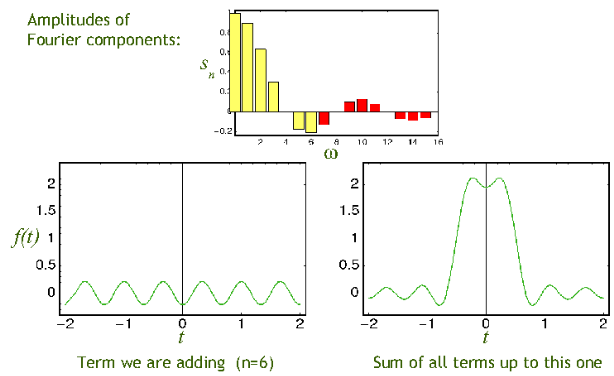


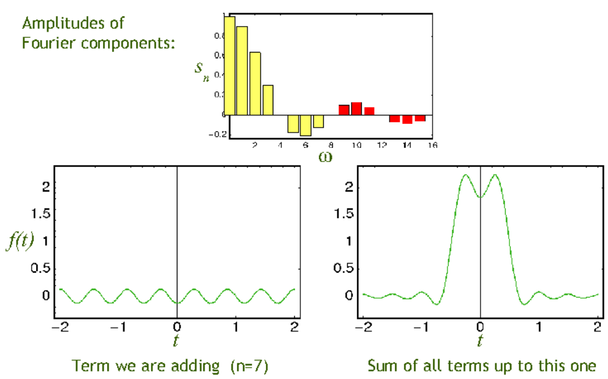
[1] The following figures are from the presentation on Fourier transform in Magnetic Resonance Imaging "The Fourier Transform and its Applications" by Branimir Vasilić. The author is grateful to Branimir for making the slides available.

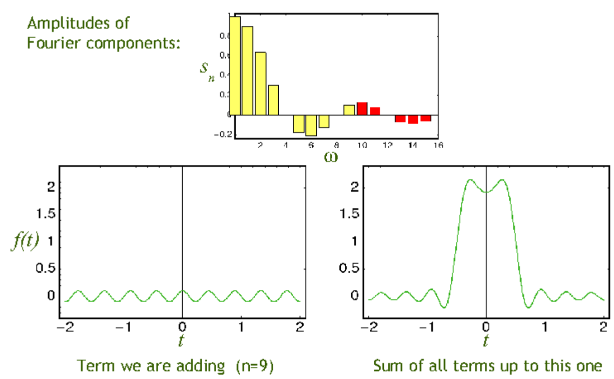

Amplitudes of
Fourier components:
s_n
ω
f(t)
t
Term we are adding (n=2)
Sum of all terms up to this one

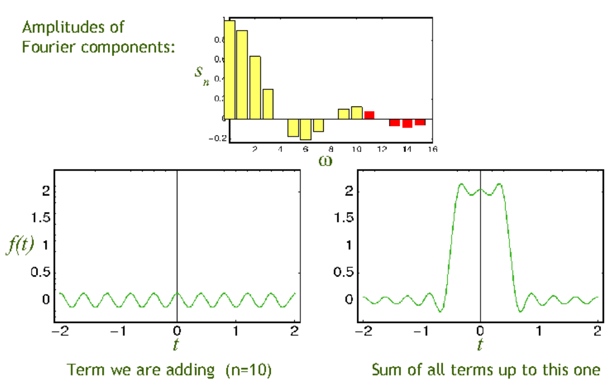

Amplitudes of
Fourier components:
s_n
ω
f(t)
t
Term we are adding (n=0)
Sum of all terms up to this one

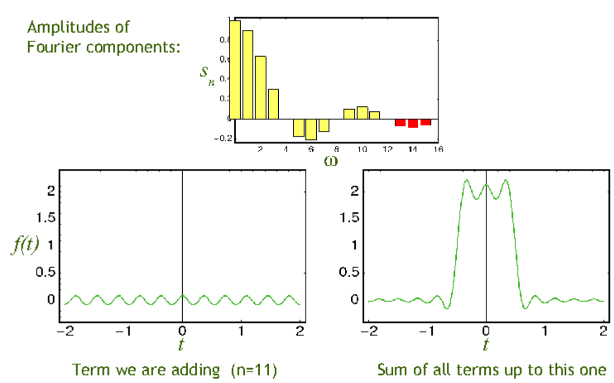

Amplitudes of
Fourier components:
s_n
ω
f(t)
t
Term we are adding (n=5)
Sum of all terms up to this one

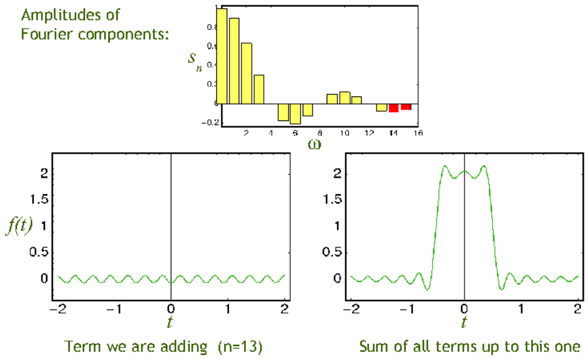

Amplitudes of
Fourier components:
s_n
ω
f(t)
t
Term we are adding (n=6)
Sum of all terms up to this one

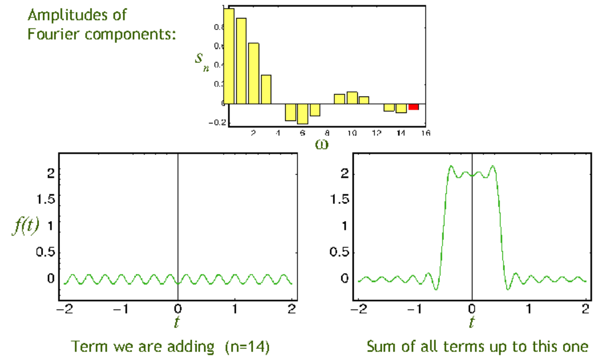

Amplitudes of
Fourier components:
s_n
ω
f(t)
t
Term we are adding (n=7)
Sum of all terms up to this one

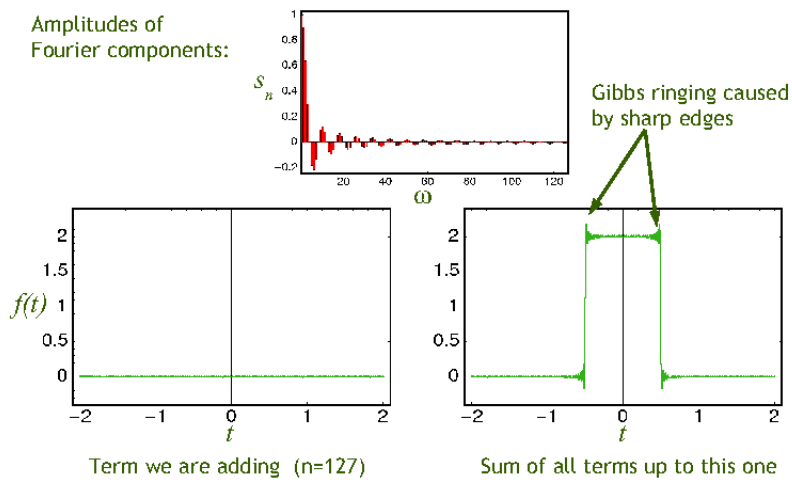

Amplitudes of
Fourier components:
s_n
ω
f(t)
t
Term we are adding (n=9)
Sum of all terms up to this one


Amplitudes of
Fourier components:
s_n
ω
f(t)
t
Term we are adding (n=10)
Sum of all terms up to this one


Amplitudes of
Fourier components:
s_n
ω
f(t)
t
Term we are adding (n=11)
Sum of all terms up to this one







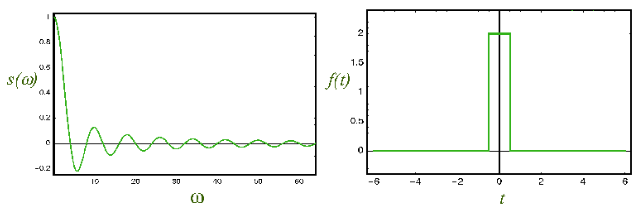
Discrete versus Continuous Functions.

In the previous examples, the function in frequency-domain was a continuous function. In applications, it is impossible to collect infinitely many infinitely dense samples. As a consequence, certain error may come from sampling just finite number of points taken over a finite interval. Thus, it is relevant to keep in mind what effects in time (resp. frequency) domain may have finite and discrete samples of frequencies (resp. time).

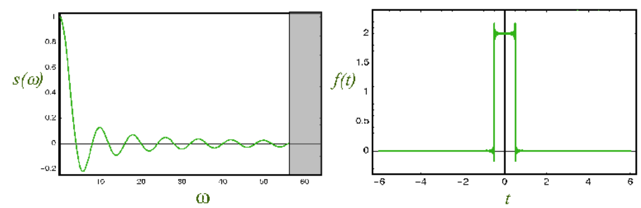
frequency-domain	function in time-domain	effects in time-domain
infinite continuous set of frequencies	non-periodic function	none
finite continuous set of frequencies	non-periodic function	Gibbs ringing, blurring
infinite number of discrete frequencies	periodic function	aliasing
finite number of discrete frequencies	periodic function	aliasing, Gibbs ringing, blurring

In the following consideration, assume that the sample consists of frequencies and that all the values lie on the graph of a sinc function. The goal is to reconstruct the boxcar function in the time-domain. The following four figures illustrate each of the four scenarios from the table.

If we sum an **infinite range of continuous frequencies** we get an exact version of the initial function *f(t)*

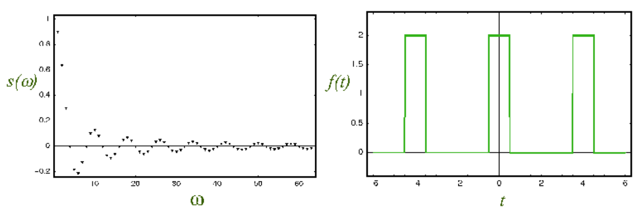


If we sum a **finite range of continous frequencies** we get an approximate version of the function *f(t)* that is **blurred** and has **Gibbs ringing**

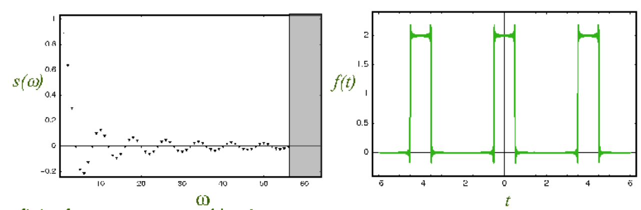


If we sum an **infinite range of discrete frequencies** we get an exact version of the function *f(t)* repeated an infinite number of times

The infinite repetition of *f(t)* is called **aliasing**



If we sum a **finite number of discrete frequencies** we get an **infinite number** of approximate copies of the function *f(t)* that are **blurred** and have **Gibbs ringing**

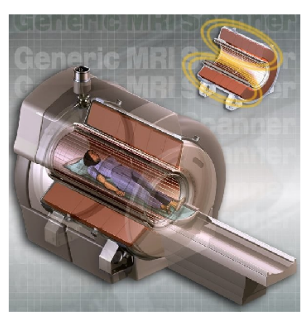


finite frequency range → **blurring**
sharp edges → **Gibbs ringing**
discrete frequencies → **aliasing**

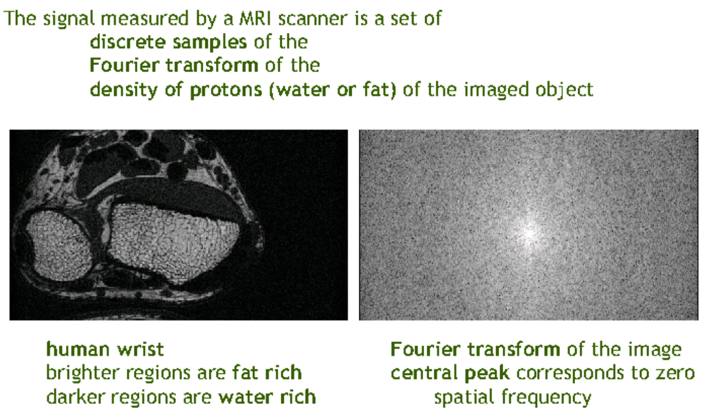
Application in Magnetic Resonance Imaging.

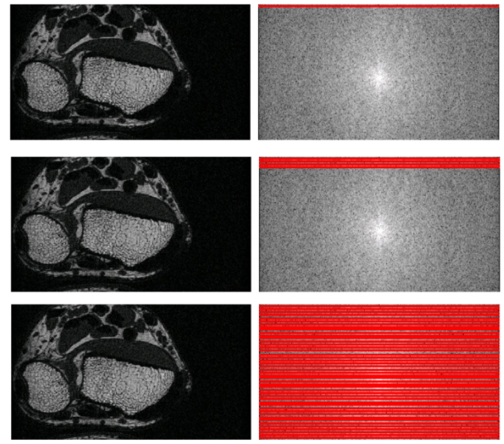
The Fourier transform is prominently used in Magnetic Resonance Imaging (MRI). The fist figure represents a typical MRI scanner. The scanner samples spacial frequencies and creates the Fourier transform of the image we would like to obtain. The inverse Fourier transform converts the measured signal (input) into the reconstructed image (output).

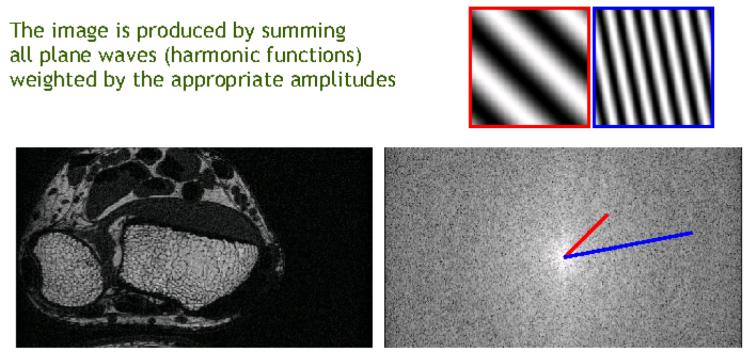
A MRI scanner consists of a large **superconducting magnet** and **coils** that generate **temporally and spatially varying magnetic fields** (no ionizing radiation)



In the following two figures, the image on the right represents the amplitude of the scanned input signal. The image on the left represents the output - the image of a human wrist, more precisely, the density of protons in a human wrist. The whiter area on the image correspond to the fat rich regions. The darker area on the image correspond to the water rich regions.





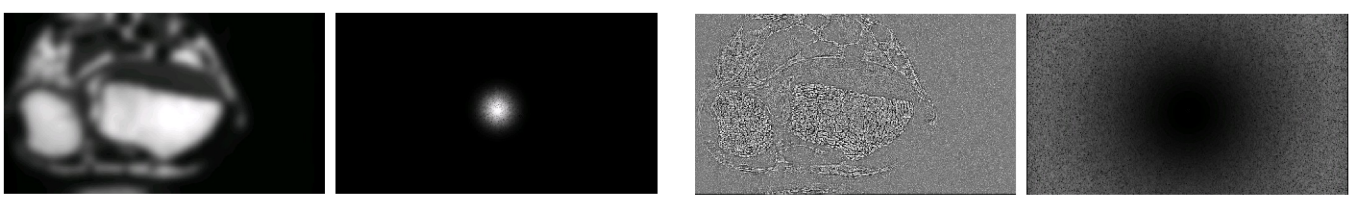


Each point on the input images corresponds to certain frequency. Two smaller figures on the right side represent the components of the inverse transforms at two highlighted frequencies. The output image is created by combining many such images - one for each sampled frequency in fact.

The last figure represent the output with high (first pair of images) and low frequencies (second pair of images) removed.

If high frequencies are removed (low-pass filter), the image becomes blurred and only shows the rough shape of the object.

If low frequencies are removed (high-pass filter), the image is sharp but intensity variations are lost.



Practice Problems.

1. Find the Fourier transform of $f(t) = e^{-t}$, $t > 0$, $f(t) = 0$ otherwise.

2. Find Fourier cosine transformation of the function from the previous problem.

3. Find cosine Fourier transform of $f(t) = 2t - 3$ for $0 < t < 3/2$, $f(x) = 0$ otherwise.

4. If

$$f_a(t) = \frac{a}{a^2+t^2}$$

the graph of f has a peak at 0. Since $f(0)=\frac{1}{a}$, the height is conversely proportional to a. The Fourier transform of f can shown to be

$$F_a(\omega) = \sqrt{\pi/2}e^{-a|\omega|}.$$

Graph $F_a(\omega)$ for several values of a and make conclusion how values of a impact the graph of F_a.

5. Solve the equation $\int_0^\infty f(t)\cos t\omega\, dt = \begin{cases} 1-\omega & 0\le\omega\le 1 \\ 0 & \omega > 1 \end{cases}$

Solutions.

1. $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t}e^{-i\omega t}dt = \frac{-1}{\sqrt{2\pi}(1+i\omega)}e^{-(1+i\omega)t}|_0^\infty = \frac{1}{\sqrt{2\pi}(1+i\omega)}$.

2. $F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-t}\cos\omega t\; dt$. Using two integration by parts with $u=e^{-t}$, one obtains that $F(\omega) = \sqrt{\frac{2}{\pi}}(\frac{1}{\omega}e^{-t}\sin\omega t|_0^\infty + \frac{1}{\omega}\int_0^\infty e^{-t}\sin\omega t\; dt) = \sqrt{\frac{2}{\pi}}(0 - \frac{1}{\omega^2}e^{-t}\cos\omega t|_0^\infty - \frac{1}{\omega^2}\int_0^\infty e^{-t}\cos\omega t\; dt) = \sqrt{\frac{2}{\pi}}(\frac{1}{\omega^2} - \frac{1}{\omega^2}\sqrt{\frac{\pi}{2}}F(\omega)$. Solving for $F(\omega)$ gives your $F(\omega)(1+\frac{1}{\omega^2}) = \sqrt{\frac{2}{\pi}}\frac{1}{\omega^2}$. Multiply by ω^2 to get $F(\omega)(\omega^2+1) = \sqrt{\frac{2}{\pi}} \Rightarrow F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\omega^2+1}$.

3. $F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{3/2}(2t-3)\cos\omega t\; dt = \sqrt{\frac{2}{\pi}}(\frac{2t-3}{\omega}\sin\omega t|_0^{3/2} - \frac{2}{\omega}\int_0^{3/2}\sin\omega t\; dt) =$
$\sqrt{\frac{2}{\pi}}(0 + \frac{2}{\omega^2}\cos\omega t|_0^{3/2}) = \sqrt{\frac{2}{\pi}}\frac{2}{\omega^2}(\cos\frac{3\omega}{2} - 1) = \frac{2\sqrt{2}}{\omega^2\sqrt{\pi}}(\cos\frac{3\omega}{2} - 1)$.

4. If a is small, then f_a has a larger peak. In this case F_a is more spread out and flattened. If a is large, f_a is spread out and the height of the peak is not large. In this case, F_a has a sharp peak and converges to zero fast.

5. Use inverse cosine Fourier transform. First multiply with $\sqrt{\frac{2}{\pi}}$ to match the definition of the transform. Then the left side is exactly the Fourier cosine transform of $f(t)$. So we can get $f(t)$ as the inverse transform of the left side of the equation multiplied by $\sqrt{\frac{2}{\pi}}$.

   $f(t) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^1(1-\omega)\cos\omega t\; d\omega = \frac{2}{\pi}(\frac{1-\omega}{t}\sin\omega t|_0^1 + \frac{1}{t}\int_0^1\sin\omega t\; dt) = \frac{2}{\pi}(0 - \frac{1}{t^2}\cos\omega t|_0^1) = \frac{2}{\pi t^2}(-\cos t + 1) = \frac{2}{\pi t^2}(1-\cos t)$.