

A

The Simplex Solution Method

The *simplex method*, is a general mathematical solution technique for solving linear programming problems. In the simplex method, the model is put into the form of a table, and then a number of mathematical steps are performed on the table. These mathematical steps in effect replicate the process in graphical analysis of moving from one extreme point on the solution boundary to another. However, unlike the graphical method, in which we could simply search through *all* the solution points to find the best one, the simplex method moves from one *better* solution to another until the best one is found, and then it stops.

The manual solution of a linear programming model using the simplex method can be a lengthy and tedious process. Years ago, manual application of the simplex method was the only means for solving a linear programming problem. Now computer solution is certainly preferred. However, knowledge of the simplex method can greatly enhance one's understanding of linear programming. Computer software programs like QM for Windows or Excel spreadsheets provide solutions to linear programming problems, but they do not convey an in-depth understanding of how those solutions are derived. To a certain extent, graphical analysis provides an understanding of the solution process, and knowledge of the simplex method further expands on that understanding. In fact, computer solutions are usually derived using the simplex method. As a result, much of the terminology and notation used in computer software comes from the simplex method. Thus, for those students of management science who desire a more in-depth knowledge of linear programming, it is beneficial to study the simplex solution method as provided here.

Converting the Model into Standard Form

The first step in solving a linear programming model manually with the simplex method is to convert the model into standard form. At the Beaver Creek Pottery Company Native American artisans produce bowls (x_1) and mugs (x_2) from labor and clay. The linear programming model is formulated as

$$\text{maximize } Z = \$40x_1 + 50x_2 \text{ (profit)}$$

subject to

$$x_1 + 2x_2 \leq 40 \text{ (labor, hr)}$$

$$4x_1 + 3x_2 \leq 120 \text{ (clay, lb)}$$

$$x_1, x_2 \geq 0$$

We convert this model into *standard form* by adding slack variables to each constraint as follows.

$$\text{maximize } Z = 40x_1 + 50x_2 + 0s_1 + 0s_2$$

subject to

$$\begin{aligned}x_1 + 2x_2 + s_1 &= 40 \\4x_1 + 3x_2 + s_2 &= 120 \\x_1, x_2, s_1, s_2 &\geq 0\end{aligned}$$

Slack variables are added to \leq constraints and represent unused resources.

The slack variables, s_1 and s_2 , represent the amount of unused labor and clay, respectively. For example, if no bowls and mugs are produced, and $x_1 = 0$ and $x_2 = 0$, then the solution to the problem is

$$\begin{aligned}x_1 + 2x_2 + s_1 &= 40 \\0 + 2(0) + s_1 &= 40 \\s_1 &= 40 \text{ hr of labor}\end{aligned}$$

and

$$\begin{aligned}4x_1 + 3x_2 + s_2 &= 120 \\4(0) + 3(0) + s_2 &= 120 \\s_2 &= 120 \text{ lb of clay}\end{aligned}$$

In other words, when we start the problem and nothing is being produced, all the resources are unused. Since unused resources contribute nothing to profit, the profit is zero.

$$\begin{aligned}Z &= \$40x_1 + 50x_2 + 0s_1 + 0s_2 \\&= 40(0) + 50(0) + 0(40) + 0(120) \\Z &= \$0\end{aligned}$$

It is at this point that we begin to apply the simplex method. The model is in the required form, with the inequality constraints converted to equations for solution with the simplex method.

The Solution of Simultaneous Equations

Once both model constraints have been transformed into equations, the equations should be solved simultaneously to determine the values of the variables at every possible solution point. However, notice that our example problem has *two* equations and *four* unknowns (i.e., two decision variables and two slack variables), a situation that makes direct simultaneous solution impossible. The simplex method alleviates this problem by assigning some of the variables a value of zero. The number of variables assigned values of zero is $n - m$, where n equals the number of variables and m equals the number of constraints (excluding the nonnegativity constraints). For this model, $n = 4$ variables and $m = 2$ constraints; therefore, two of the variables are assigned a value of zero (i.e., $4 - 2 = 2$).

For example, letting $x_1 = 0$ and $s_1 = 0$ results in the following set of equations.

$$\begin{aligned}x_1 + 2x_2 + s_1 &= 40 \\4x_1 + 3x_2 + s_2 &= 120\end{aligned}$$

and

$$\begin{aligned}0 + 2x_2 + 0 &= 40 \\0 + 3x_2 + s_2 &= 120\end{aligned}$$

First, solve for x_2 in the first equation:

$$\begin{aligned}2x_2 &= 40 \\x_2 &= 20\end{aligned}$$

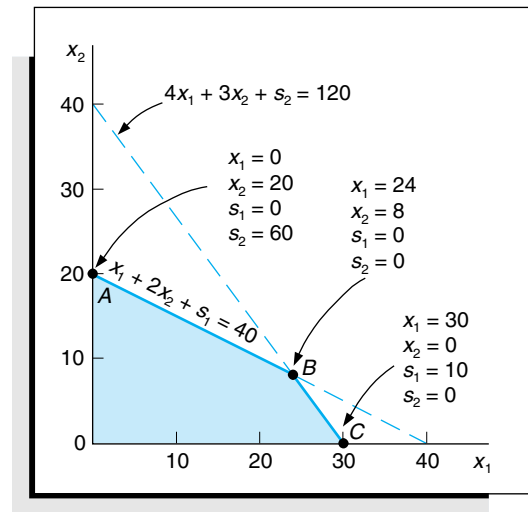
Then, solve for s_2 in the second equation:

$$\begin{aligned} 3x_2 + s_2 &= 120 \\ 3(20) + s_2 &= 120 \\ s_2 &= 60 \end{aligned}$$

A **basic feasible solution** satisfies the model constraints and has the same number of variables with non-negative values as there are constraints.

This solution corresponds with point A in Figure A-1. The graph in Figure A-1 shows that at point A, $x_1 = 0$, $x_2 = 20$, $s_1 = 0$, and $s_2 = 60$, the exact solution obtained by solving simultaneous equations. This solution is referred to as a *basic feasible solution*. A feasible solution is any solution that satisfies the constraints. A *basic feasible solution* satisfies the constraints and contains as many variables with nonnegative values as there are model constraints—that is, m variables with nonnegative values and $n - m$ values set equal to zero. Typically, the m variables have positive nonzero solution values; however, when one of the m variables equals zero, the basic feasible solution is said to be *degenerate*. (The topic of *degeneracy* will be discussed at a later point in this module.)

Figure A-1
Solutions at points A, B, and C



Consider a second example where $x_2 = 0$ and $s_2 = 0$. These values result in the following set of equations.

$$\begin{aligned} x_1 + 2x_2 + s_1 &= 40 \\ 4x_1 + 3x_2 + s_2 &= 120 \end{aligned}$$

and

$$\begin{aligned} x_1 + 0 + s_1 &= 40 \\ 4x_1 + 0 + 0 &= 120 \end{aligned}$$

Solve for x_1 :

$$\begin{aligned} 4x_1 &= 120 \\ x_1 &= 30 \end{aligned}$$

Then solve for s_1 :

$$\begin{aligned} 30 + s_1 &= 40 \\ s_1 &= 10 \end{aligned}$$

This basic feasible solution corresponds to point C in Figure A-1, where $x_1 = 30$, $x_2 = 0$, $s_1 = 10$, and $s_2 = 0$.

Finally, consider an example where $s_1 = 0$ and $s_2 = 0$. These values result in the following set of equations.

$$\begin{aligned}x_1 + 2x_2 + s_1 &= 40 \\4x_1 + 3x_2 + s_2 &= 120\end{aligned}$$

and

$$\begin{aligned}x_1 + 2x_2 + 0 &= 40 \\4x_1 + 3x_2 + 0 &= 120\end{aligned}$$

Row operations are used to solve simultaneous equations where equations are multiplied by constants and added or subtracted from each other.

These equations can be solved using *row operations*. In row operations, the equations can be multiplied by constant values and then added or subtracted from each other without changing the values of the decision variables. First, multiply the top equation by 4 to get

$$4x_1 + 8x_2 = 160$$

and then subtract the second equation:

$$\begin{array}{r}4x_1 + 8x_2 = 160 \\-4x_1 - 3x_2 = -120 \\ \hline 5x_2 = 40 \\ x_2 = 8\end{array}$$

Next, substitute this value of x_2 into either one of the constraints.

$$\begin{aligned}x_1 + 2(8) &= 40 \\ x_1 &= 24\end{aligned}$$

This solution corresponds to point B on the graph, where $x_1 = 24$, $x_2 = 8$, $s_1 = 0$, and $s_2 = 0$, which is the optimal solution point.

All three of these example solutions meet our definition of *basic feasible solutions*. However, two specific questions are raised by the identification of these solutions.

1. In each example, how was it known which variables to set equal to zero?
2. How is the optimal solution identified?

The answers to both of these questions can be found by using the simplex method. The simplex method is a set of mathematical steps that determines at each step which variables should equal zero and when an optimal solution has been reached.

The Simplex Method

The **simplex method** is a set of mathematical steps for solving a linear programming problem carried out in a table called a **simplex tableau**.

The steps of the simplex method are carried out within the framework of a table, or *tableau*. The tableau organizes the model into a form that makes applying the mathematical steps easier. The Beaver Creek Pottery Company example will be used again to demonstrate the simplex tableau and method.

$$\text{maximize } Z = \$40x_1 + 50x_2 + 0s_1 + 0s_2$$

subject to

$$\begin{aligned}x_1 + 2x_2 + s_1 &= 40 \text{ hr} \\4x_1 + 3x_2 + s_2 &= 120 \text{ lb} \\x_1, x_2, s_1, s_2 &\geq 0\end{aligned}$$

The initial simplex tableau for this model, with the various column and row headings, is shown in Table A-1.

Table A-1
The Simplex Tableau

c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2
	z_j					
	$c_j - z_j$					

The first step in filling in Table A-1 is to record the model variables along the second row from the top. The two decision variables are listed first, in order of their subscript magnitude, followed by the slack variables, also listed in order of their subscript magnitude. This step produces the row with $x_1, x_2, s_1,$ and s_2 in Table A-1.

The basic feasible solution in the initial simplex tableau is the origin where all decision variables equal zero.

The next step is to determine a basic feasible solution. In other words, which two variables will form the basic feasible solution and which will be assigned a value of zero? Instead of arbitrarily selecting a point (as we did with points A, B, and C in the previous section), the simplex method selects the origin as the initial basic feasible solution because the values of the decision variables at the origin are always known in all linear programming problems. At that point $x_1 = 0$ and $x_2 = 0$; thus, the variables in the basic feasible solution are s_1 and s_2 .

$$\begin{aligned} x_1 + 2x_2 + s_1 &= 40 \\ 0 + 2(0) + s_1 &= 40 \\ s_1 &= 40 \text{ hr} \end{aligned}$$

and

$$\begin{aligned} 4x_1 + 3x_2 + s_2 &= 120 \\ 4(0) + 3(0) + s_2 &= 120 \\ s_2 &= 120 \text{ lb} \end{aligned}$$

At the initial basic feasible solution at the origin, only slack variables have a value greater than zero.

In other words, at the origin, where there is no production, all resources are slack, or unused. The variables s_1 and s_2 , which form the initial basic feasible solution, are listed in Table A-2 under the column “Basic Variables,” and their respective values, 40 and 120, are listed under the column “Quantity.”

Table A-2
The Basic Feasible Solution

c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2
	s_1	40				
	s_2	120				
	z_j					
	$c_j - z_j$					

The quantity column values are the solution values for the variables in the basic feasible solution.

The number of rows in a tableau is equal to the number of constraints plus four.

The number of columns in a tableau is equal to the number of variables (including slacks, etc.) plus three.

The c_j values are the contribution to profit (or cost) for each variable.

The initial simplex tableau always begins with the solution at the origin, where x_1 and x_2 equal zero. Thus, the basic variables at the origin are the slack variables, s_1 and s_2 . Since the quantity values in the initial solution always appear as the right-hand-side values of the constraint equations, they can be read directly from the original constraint equations.

The top two rows and bottom two rows are standard for all tableaus; however, the number of middle rows is equivalent to the number of constraints in the model. For example, this problem has two constraints; therefore, it has two middle rows corresponding to s_1 and s_2 . (Recall that n variables minus m constraints equals the number of variables in the problem with values of zero. This also means that the number of basic variables with values other than zero will be equal to m constraints.)

Similarly, the three columns on the left side of the tableau are standard, and the remaining columns are equivalent to the number of variables. Since there are four variables in this model, there are four columns on the right of the tableau, corresponding to $x_1, x_2, s_1,$ and s_2 .

The next step is to fill in the c_j values, which are the objective function coefficients, representing the contribution to profit (or cost) for each variable x_j or s_j in the objective function. Across the top row the c_j values 40, 50, 0, and 0 are inserted for each variable in the model, as shown in Table A-3.

Table A-3
The Simplex Tableau with c_j Values

c_j	Basic Variables	Quantity	40 x_1	50 x_2	0 s_1	0 s_2
0	s_1	40				
0	s_2	120				
	z_j					
	$c_j - z_j$					

The values for c_j on the left side of the tableau are the contributions to profit of only those variables in the basic feasible solution, in this case s_1 and s_2 . These values are inserted at this location in the tableau so that they can be used later to compute the values in the z_j row.

The columns under each variable (i.e., $x_1, x_2, s_1,$ and s_2) are filled in with the coefficients of the decision variables and slack variables in the model constraint equations. The s_1 row represents the first model constraint; thus, the coefficient for x_1 is 1, the coefficient for x_2 is 2, the coefficient for s_1 is 1, and the coefficient for s_2 is 0. The values in the s_2 row are the second constraint equation coefficients, 4, 3, 0, and 1, as shown in Table A-4.

Table A-4
The Simplex Tableau with Model Constraint Coefficients

c_j	Basic Variables	Quantity	40 x_1	50 x_2	0 s_1	0 s_2
0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	z_j					
	$c_j - z_j$					

This completes the process of filling in the initial simplex tableau. The remaining values in the z_j and $c_j - z_j$ rows, as well as subsequent tableau values, are computed mathematically using simplex formulas.

The following list summarizes the steps of the simplex method (for a maximization model) that have been presented so far.

1. First, transform all inequalities to equations by adding slack variables.
2. Develop a simplex tableau with the number of columns equaling the number of variables plus three, and the number of rows equaling the number of constraints plus four.
3. Set up table headings that list the model decision variables and slack variables.
4. Insert the initial basic feasible solution, which are the slack variables and their quantity values.
5. Assign c_j values for the model variables in the top row and the basic feasible solution variables on the left side.
6. Insert the model constraint coefficients into the body of the table.

Computing the z_j and $c_j - z_j$ Rows

The z_j row values are computed by multiplying the c_j column values by the variable column values and summing.

So far the simplex tableau has been set up using values taken directly from the model. From this point on the values are determined by computation. First, the values in the z_j row are computed by multiplying each c_j column value (on the left side) by each column value under Quantity, x_1 , x_2 , s_1 , and s_2 and then summing each of these sets of values. The z_j values are shown in Table A-5.

Table A-5
The Simplex Tableau with z_j Row Values

c_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	z_j	0	0	0	0	0
	$c_j - z_j$					

For example, the value in the z_j row under the quantity column is found as follows.

$$\begin{aligned}
 &c_j \quad \text{Quantity} \\
 &0 \times 40 = 0 \\
 &0 \times 120 = 0 \\
 &z_q = 0
 \end{aligned}$$

The value in the z_j row under the x_1 column is found similarly.

$$\begin{aligned}
 &c_j \quad x_1 \\
 &0 \times 1 = 0 \\
 &0 \times 4 = 0 \\
 &z_j = 0
 \end{aligned}$$

All of the other z_j row values for this tableau will be zero when they are computed using this formula.

Now the $c_j - z_j$ row is computed by subtracting the z_j row values from the c_j (top) row values. For example, in the x_1 column the $c_j - z_j$ row value is computed as $40 - 0 = 40$.

The simplex method works by moving from one solution (extreme) point to an adjacent point until it locates the best solution.

This value as well as other $c_j - z_j$ values are shown in Table A-6, which is the complete initial simplex tableau with all values filled in. This tableau represents the solution at the origin, where $x_1 = 0$, $x_2 = 0$, $s_1 = 40$, and $s_2 = 120$. The profit represented by this solution (i.e., the Z value) is given in the z_j row under the quantity column—0 in Table A-6. This solution is obviously not optimal because no profit is being made. Thus, we want to move to a solution point that will give a *better* solution. In other words, we want to produce either some bowls (x_1) or some mugs (x_2). One of the *nonbasic* variables (i.e., variables not in the present basic feasible solution) will enter the solution and become basic.

Table A-6
The Complete Initial Simplex Tableau

c_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	z_j	0	0	0	0	0
	$c_j - z_j$		40	50	0	0

The Entering Nonbasic Variable

As an example, suppose the pottery company decides to produce some bowls. With this decision x_1 will become a basic variable. For every unit of x_1 (i.e., each bowl) produced, profit will be increased by \$40 because that is the profit contribution of a bowl. However, when a bowl (x_1) is produced, some previously unused resources will be used. For example, if

$$x_1 = 1$$

then

$$\begin{aligned} x_1 + 2x_2 + s_1 &= 40 \text{ hr of labor} \\ 1 + 2(0) + s_1 &= 40 \\ s_1 &= 39 \text{ hr of labor} \end{aligned}$$

and

$$\begin{aligned} 4x_1 + 3x_2 + s_2 &= 120 \text{ lb of clay} \\ 4(1) + 3(0) + s_2 &= 120 \\ s_2 &= 116 \text{ lb of clay} \end{aligned}$$

In the labor constraint we see that, with the production of one bowl, the amount of slack, or unused, labor is *decreased* by 1 hour. In the clay constraint the amount of slack is *decreased* by 4 pounds. Substituting these increases (for x_1) and decreases (for slack) into the objective function gives

$$\begin{aligned} Z &= 40(1) + 50(0) + 0(-1) + 0(-4) \\ Z &= \$40 \end{aligned}$$

The variable with the largest positive $c_j - z_j$ value is the entering variable.

The first part of this objective function relationship represents the values in the c_j row; the second part represents the values in the z_j row. The function expresses the fact that to produce some bowls, we must give up some of the profit already earned from the items they replace. In this case the production of bowls replaced only slack, so no profit was lost. In general, the $c_j - z_j$ row values represent the net *increase per unit of entering a nonbasic*

variable into the basic solution. Naturally, we want to make as much money as possible, because the objective is to maximize profit. Therefore, we enter the variable that will give the greatest net increase in profit per unit. From Table A-7, we select variable x_2 as the entering basic variable because it has the greatest net increase in profit per unit, \$50 — the highest positive value in the $c_j - z_j$ row.

Table A-7
Selection of the Entering Basic Variable

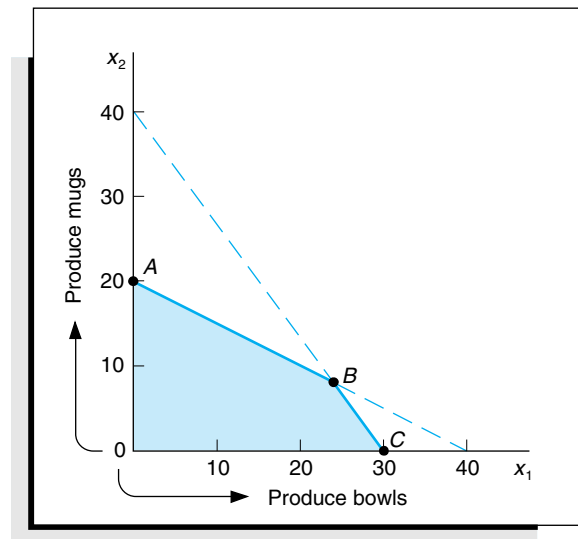
c_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	z_j	0	0	0	0	0
	$c_j - z_j$		40	50	0	0

The **pivot column** is the column corresponding to the entering variable.

The x_2 column, highlighted in Table A-7, is referred to as the *pivot column*. (The operations used to solve simultaneous equations are often referred to in mathematical terminology as *pivot operations*.)

The selection of the entering basic variable is also demonstrated by the graph in Figure A-2. At the origin nothing is produced. In the simplex method we move from one solution point to an *adjacent* point (i.e., one variable in the basic feasible solution is replaced with a variable that was previously zero). In Figure A-2 we can move along either the x_1 axis or the x_2 axis in order to seek a better solution. Because an increase in x_2 will result in a greater profit, we choose x_2 .

Figure A-2
Selection of which item to produce — the entering basic variable



The Leaving Basic Variable

Since each basic feasible solution contains only two variables with nonzero values, one of the two basic variables present, s_1 or s_2 , will have to leave the solution and become zero. Since we have decided to produce mugs (x_2), we want to produce as many as possible or, in other words, as many as our resources will allow. First, in the labor constraint we will use all

the labor to make mugs (because no bowls are to be produced, $x_1 = 0$; and because we will use all the labor possible and $s_1 =$ unused labor resources, $s_1 = 0$ also).

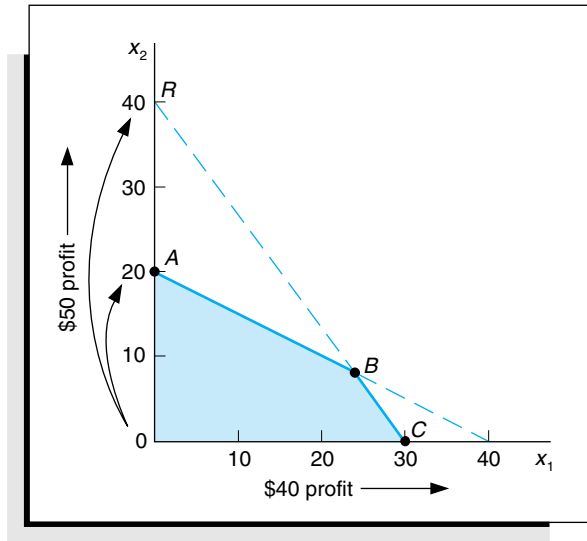
$$\begin{aligned} 1x_1 + 2x_2 + s_1 &= 40 \text{ hr} \\ 1(0) + 2x_2 + 0 &= 40 \\ x_2 &= \frac{40 \text{ hr}}{2 \text{ hr/mug}} \\ &= 20 \text{ mugs} \end{aligned}$$

In other words, enough labor is available to produce 20 mugs. Next, perform the same analysis on the constraint for clay.

$$\begin{aligned} 4x_1 + 3x_2 + s_2 &= 120 \text{ lb} \\ 4(0) + 3x_2 + 0 &= 120 \\ x_2 &= \frac{120 \text{ lb}}{3 \text{ lb/mug}} \\ &= 40 \text{ mugs} \end{aligned}$$

This indicates that there is enough clay to produce 40 mugs. But there is enough labor to produce only 20 mugs. We are limited to the production of only 20 mugs because we do not have enough labor to produce any more than that. This analysis is shown graphically in Figure A-3.

Figure A-3
Determination of the basic feasible solution point



Because we are moving out the x_2 axis, we can move from the origin to either point A or point R. We select point A because it is the *most constrained* and thus feasible, whereas point R is infeasible.

This analysis is performed in the simplex method by dividing the quantity values of the basic solution variables by the pivot column values. For this tableau,

The leaving variable is determined by dividing the quantity values by the pivot column values and selecting the minimum possible value or zero.

Basic Variables	Quantity	x_2
s_1	40	$\div 2 = 20$, the leaving basic variable
s_2	120	$\div 3 = 40$

The leaving basic variable is the variable that corresponds to the minimum nonnegative quotient, which in this case is 20. (Note that a value of zero would qualify as the minimum quotient and would be the choice for the leaving variable.) Therefore, s_1 is the leaving variable. (At point A in Figure A-3, s_1 equals zero because all the labor is used to make the 20 mugs.) The s_1 row, highlighted in Table A-8, is also referred to as the *pivot row*.

The **pivot row** is the row corresponding to the leaving variable.

Table A-8
Pivot Column, Pivot Row, and Pivot Number

			40	50	0	0
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2
0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	z_j	0	0	0	0	0
	$c_j - z_j$		40	50	0	0

The value of 2 at the intersection of the pivot row and the pivot column is called the *pivot number*. The pivot number, row, and column are all instrumental in developing the next tableau. We are now ready to proceed to the second simplex tableau and a *better* solution.

The **pivot number** is the number at the intersection of the pivot column and row.

Developing a New Tableau

Table A-9 shows the second simplex tableau with the new basic feasible solution variables of x_2 and s_2 and their corresponding c_j values.

Table A-9
The Basic Variables and c_j Values for the Second Simplex Tableau

			40	50	0	0
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2
50	x_2					
0	s_2					
	z_j					
	$c_j - z_j$					

Computing the new tableau pivot row values.

The various row values in the second tableau are computed using several simplex formulas. First, the x_2 row, called the *new tableau pivot row*, is computed by dividing every value in the pivot row of the first (old) tableau by the pivot number. The formula for these computations is

$$\text{new tableau pivot row values} = \frac{\text{old tableau pivot row values}}{\text{pivot number}}$$

The new row values are shown in Table A-10.

Computing all remaining row values.

To compute all remaining row values (in this case there is only one other row), another formula is used.

Table A-10
Computation of the New Pivot
Row Values

c_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
50	x_2	20	1/2	1	1/2	0
0	s_2					
	z_j					
	$c_j - z_j$					

$$\text{new tableau row values} = \text{old tableau row values} - \left(\begin{matrix} \text{corresponding} & \text{corresponding} \\ \text{coefficients in} & \text{new tableau} \\ \text{pivot column} & \text{pivot row value} \end{matrix} \right)$$

Thus, this formula requires the use of both the old tableau and the new one. The s_2 row values are computed in Table A-11.

Table A-11
Computation of New s_2
Row Values

Column	Old Tableau Row Value	–	Corresponding Coefficients in Pivot Column	×	New Tableau Pivot Row Value	=	New Tableau Row Value
Quantity	120	–	(3	×	20)	=	60
x_1	4	–	(3	×	1/2)	=	5/2
x_2	3	–	(3	×	1)	=	0
s_1	0	–	(3	×	1/2)	=	–3/2
s_2	1	–	(3	×	0)	=	1

These values have been inserted in the simplex tableau in Table A-12.

This solution corresponds to point A in the graph of this model in Figure A-3. The solution at this point is $x_1 = 0, x_2 = 20, s_1 = 0, s_2 = 60$. In other words, 20 mugs are produced and 60 pounds of clay are left unused. No bowls are produced and no labor hours remain unused.

Table A-12
The Second Simplex Tableau
with Row Values

c_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
50	x_2	20	1/2	1	1/2	0
0	s_2	60	5/2	0	–3/2	1
	z_j					
	$c_j - z_j$					

The second simplex tableau is completed by computing the z_j and $c_j - z_j$ row values the same way they were computed in the first tableau. The z_j row is computed by summing the products of the c_j column and all other column values.

Time Out

for George B. Dantzig

After developing the simplex method for solving linear programming problems, George Dantzig needed a good problem to test it on. The problem he selected was the “diet problem” formulated in 1945 by Nobel economist George Stigler. This problem was to determine an adequate nutritional diet at minimum cost (which was an important military and civilian issue during World War II). Formulated as a linear programming

model, the diet problem consisted of 77 unknowns and 9 equations. It took 9 clerks using hand-operated (mechanical) desk calculators 120 man-days to obtain the optimal simplex solution: a diet consisting primarily of wheat flour, cabbage, and dried navy beans that cost \$39.69 per year (in 1939 prices). The solution developed by Stigler using his own numerical method was only 24 cents more than the optimal solution.

Column

Quantity $z_j = (50)(20) + (0)(60) = 1000$
 x_1 $z_1 = (50)(1/2) + (0)(5/2) = 25$
 x_2 $z_2 = (50)(1) + (0)(0) = 50$
 s_1 $z_3 = (50)(1/2) + (0)(-3/2) = 25$
 s_2 $z_4 = (50)(0) + (0)(1) = 0$

The z_j row values and the $c_j - z_j$ row values are added to the tableau to give the completed second simplex tableau shown in Table A-13. The value of 1,000 in the z_j row is the value of the objective function (profit) for this basic feasible solution.

Table A-13
The Completed Second Simplex Tableau

			40	50	0	0
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2
50	x_2	20	1/2	1	1/2	0
0	s_2	60	5/2	0	-3/2	1
	z_j	1,000	25	50	25	0
	$c_j - z_j$		15	0	-25	0

Each tableau is the same as performing row operations for a set of simultaneous equations.

The computational steps that we followed to derive the second tableau in effect accomplish the same thing as row operations in the solution of simultaneous equations. These same steps are used to derive each subsequent tableau, called iterations.

The Optimal Simplex Tableau

The steps that we followed to derive the second simplex tableau are repeated to develop the third tableau. First, the pivot column or entering basic variable is determined. Because 15 in the $c_j - z_j$ row represents the greatest positive net increase in profit, x_1 becomes the entering nonbasic variable. Dividing the pivot column values into the values in the quantity column indicates that s_2 is the leaving basic variable and corresponds to the pivot row. The pivot row, pivot column, and pivot number are indicated in Table A-14.

At this point you might be wondering why the net increase in profit per bowl (x_1) is \$15 rather than the original profit of \$40. It is because the production of bowls (x_1) will require some of the resources previously used to produce mugs (x_2) only. Producing some bowls means not producing as many mugs; thus, we are giving up some of the profit gained from producing mugs to gain even more by producing bowls. This difference is the net increase of \$15.

Table A-14
The Pivot Row, Pivot Column,
and Pivot Number

c_j	Basic Variables	Quantity	40 x_1	50 x_2	0 s_1	0 s_2
50	x_2	20	1/2	1	1/2	0
0	s_2	60	5/2	0	-3/2	1
	z_j	1,000	25	50	25	0
	$c_j - z_j$		15	0	-25	0

The new tableau pivot row (x_1) in the third simplex tableau is computed using the same formula used previously. Thus, all old pivot row values are divided through by 5/2, the pivot number. These values are shown in Table A-16. The values for the other row (x_2) are computed as shown in Table A-15.

Table A-15
Computation of the x_2 Row for
the Third Simplex Tableau

Column	Old Tableau Row Value	-	Corresponding Coefficients in Pivot Column	×	New Tableau Pivot Row Value	=	New Tableau Row Value
Quantity	20	-	(1/2	×	24)	=	8
x_1	1/2	-	(1/2	×	1)	=	0
x_2	1	-	(1/2	×	0)	=	1
s_1	1/2	-	(1/2	×	-3/5)	=	4/5
s_2	0	-	(1/2	×	2/5)	=	-1/5

These new row values, as well as the new z_j row and $c_j - z_j$ row, are shown in the completed third simplex tableau in Table A-16.

Table A-16
The Completed Third Simplex
Tableau

c_j	Basic Variables	Quantity	40 x_1	50 x_2	0 s_1	0 s_2
50	x_2	8	0	1	4/5	-1/5
40	x_1	24	1	0	-3/5	2/5
	z_j	1,360	40	50	16	6
	$c_j - z_j$		0	0	-16	-6

The solution is optimal when all $c_j - z_j$ values ≤ 0 .

Observing the $c_j - z_j$ row to determine the entering variable, we see that a nonbasic variable would not result in a positive net increase in profit, as all values in the $c_j - z_j$ row are zero or negative. This means that the optimal solution has been reached. The solution is

$$\begin{aligned}
 x_1 &= 24 \text{ bowls} \\
 x_2 &= 8 \text{ mugs} \\
 Z &= \$1,360 \text{ profit}
 \end{aligned}$$

which corresponds to point *B* in Figure A-1.

The simplex method does not guarantee integer solutions.

An additional comment should be made regarding simplex solutions in general. Although this solution resulted in *integer* values for the variables (i.e., 24 and 8), it is possible to get a fractional solution for decision variables even though the variables reflect items that should be integers, such as airplanes, television sets, bowls, and mugs. To apply the simplex method, one must accept this limitation.

Summary of the Simplex Method

The simplex method demonstrated in the previous section consists of the following steps.

1. Transform the model constraint inequalities into equations.
2. Set up the initial tableau for the basic feasible solution at the origin and compute the z_j and $c_j - z_j$ row values.
3. Determine the pivot column (entering nonbasic solution variable) by selecting the column with the highest positive value in the $c_j - z_j$ row.
4. Determine the pivot row (leaving basic solution variable) by dividing the quantity column values by the pivot column values and selecting the row with the minimum nonnegative quotient.
5. Compute the new pivot row values using the formula

$$\text{new tableau pivot row values} = \frac{\text{old tableau pivot row values}}{\text{pivot number}}$$

6. Compute all other row values using the formula

$$\text{new tableau row values} = \text{old tableau row values} - \left(\begin{array}{cc} \text{corresponding} & \text{corresponding} \\ \text{coefficients in} & \text{new tableau} \\ \text{pivot Column} & \text{pivot row values} \end{array} \right) \times$$

7. Compute the new z_j and $c_j - z_j$ rows.
8. Determine whether or not the new solution is optimal by checking the $c_j - z_j$ row. If all $c_j - z_j$ row values are zero or negative, the solution is optimal. If a positive value exists, return to step 3 and repeat the simplex steps.

Simplex Solution of a Minimization Problem

In the previous section the simplex method for solving linear programming problems was demonstrated for a maximization problem. In general, the steps of the simplex method outlined at the end of this section are used for any type of linear programming problem. However, a minimization problem requires a few changes in the normal simplex process, which we will discuss in this section.

In addition, several exceptions to the typical linear programming problem will be presented later in this module. These include problems with mixed constraints ($=$, \leq , and \geq); problems with more than one optimal solution, no feasible solution, or an unbounded solution; problems with a tie for the pivot column; problems with a tie for the pivot row; and problems with constraints with negative quantity values. None of these kinds of problems require changes in the simplex method. They are basically unusual results in individual simplex tableaus that the reader should know how to interpret and work with.

Standard Form of a Minimization Model

Consider the following linear programming model for a farmer purchasing fertilizer.

$$\text{minimize } Z = \$6x_1 + 3x_2$$

subject to

$$2x_1 + 4x_2 \geq 16 \text{ lb of nitrogen}$$

$$4x_1 + 3x_2 \geq 24 \text{ lb of phosphate}$$

where

x_1 = bags of Super-gro fertilizer

x_2 = bags of Crop-quick fertilizer

Z = farmer's total cost (\$) of purchasing fertilizer

This model is transformed into standard form by subtracting surplus variables from the two \geq constraints as follows.

$$\text{minimize } Z = 6x_1 + 3x_2 + 0s_1 + 0s_2$$

subject to

$$2x_1 + 4x_2 - s_1 = 16$$

$$4x_1 + 3x_2 - s_1 = 24$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The surplus variables represent the extra amount of nitrogen and phosphate that exceeded the minimum requirements specified in the constraints.

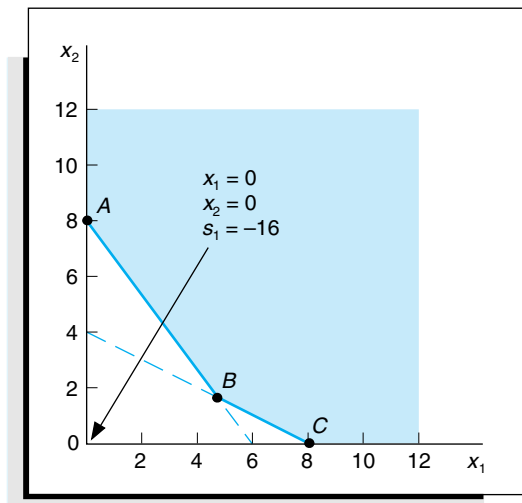
However, the simplex method requires that the initial basic feasible solution be at the origin, where $x_1 = 0$ and $x_2 = 0$. Testing these solution values, we have

$$\begin{aligned} 2x_1 + 4x_2 - s_1 &= 16 \\ 2(0) + 4(0) - s_1 &= 16 \\ s_1 &= -16 \end{aligned}$$

Transforming a model into standard form by subtracting surplus variables will not work in the simplex method.

The idea of “negative excess pounds of nitrogen” is illogical and violates the nonnegativity restriction of linear programming. The reason the surplus variable does not work is shown in Figure A-4. The solution at the origin is outside the feasible solution space.

Figure A-4
Graph of the fertilizer example



An **artificial variable** allows for an initial basic feasible solution at the origin, but it has no real meaning.

To alleviate this difficulty and get a solution at the origin, we add an *artificial variable* (A_1) to the constraint equation,

$$2x_1 + 4x_2 - s_1 + A_1 = 16$$

The artificial variable, A_1 , does not have a meaning as a slack variable or a surplus variable does. It is inserted into the equation simply to give a positive solution at the origin; we are artificially creating a solution.

$$\begin{aligned} 2x_1 + 4x_2 - s_1 + A_1 &= 16 \\ 2(0) + 4(0) - 0 + A_1 &= 16 \\ A_1 &= 16 \end{aligned}$$

The artificial variable is somewhat analogous to a booster rocket — its purpose is to get us off the ground; but once we get started, it has no real use and thus is discarded. The artificial solution helps get the simplex process started, but we do not want it to end up in the optimal solution, because it has no real meaning.

When a surplus variable is subtracted and an artificial variable is added, the phosphate constraint becomes

$$4x_1 + 3x_2 - s_2 + A_2 = 24$$

The effect of surplus and artificial variables on the objective function must now be considered. Like a slack variable, a surplus variable has no effect on the objective function in terms of increasing or decreasing cost. For example, a surplus of 24 pounds of nitrogen does not contribute to the cost of the objective function, because the cost is determined solely by the number of bags of fertilizer purchased (i.e., the values of x_1 and x_2). Thus, a coefficient of 0 is assigned to each surplus variable in the objective function.

By assigning a “cost” of \$0 to each surplus variable, we are not prohibiting it from being in the final optimal solution. It would be quite realistic to have a final solution that showed some surplus nitrogen or phosphate. Likewise, assigning a cost of \$0 to an artificial variable in the objective function would not prohibit it from being in the final optimal solution. However, if the artificial variable appeared in the solution, it would render the final solution meaningless. Therefore, we must ensure that an artificial variable is *not* in the final solution.

Artificial variables are assigned a large cost in the objective function to eliminate them from the final solution.

As previously noted, the presence of a particular variable in the final solution is based on its relative profit or cost. For example, if a bag of Super-gro costs \$600 instead of \$6 and Crop-quick stayed at \$3, it is doubtful that the farmer would purchase Super-gro (i.e., x_1 would not be in the solution). Thus, we can prohibit a variable from being in the final solution by assigning it a very *large cost*. Rather than assigning a dollar cost to an artificial variable, we will assign a value of M , which represents a large positive cost (say, \$1,000,000). This operation produces the following objective function for our example:

$$\text{minimize } Z = 6x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$$

The completely transformed minimization model can now be summarized as

$$\text{minimize } Z = 6x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$$

subject to

$$\begin{aligned} 2x_1 + 4x_2 - s_1 + A_1 &= 16 \\ 4x_1 + 3x_2 - s_2 + A_2 &= 24 \\ x_1, x_2, s_1, s_2, A_1, A_2 &\geq 0 \end{aligned}$$

The Simplex Tableau for a Minimization Problem

The $c_j - z_j$ row is changed to $z_j - c_j$ in the simplex tableau for a minimization problem.

Artificial variables are always included as part of the initial basic feasible solution when they exist.

The initial simplex tableau for a minimization model is developed the same way as one for a maximization model, except for one small difference. Rather than computing $c_j - z_j$ in the bottom row of the tableau, we compute $z_j - c_j$, which represents the *net per unit decrease in cost*, and the largest positive value is selected as the entering variable and pivot column. (An alternative would be to leave the bottom row as $c_j - z_j$ and select the largest *negative* value as the pivot column. However, to maintain a consistent rule for selecting the pivot column, we will use $z_j - c_j$.)

The initial simplex tableau for this model is shown in Table A-17. Notice that A_1 and A_2 form the initial solution at the origin, because that was the reason for inserting them in the first place—to get a solution at the origin. This is not a basic feasible solution, since the origin is not in the feasible solution area, as shown in Figure A-4. As indicated previously, it is an artificially created solution. However, the simplex process will move toward feasibility in subsequent tableaus. Note that across the top the decision variables are listed first, then surplus variables, and finally artificial variables.

Table A-17
The Initial Simplex Tableau

c_j	Basic Variables	Quantity	6	3	0	0	M	M
			x_1	x_2	s_1	s_2	A_1	A_2
M	A_1	16	2	4	-1	0	1	0
M	A_2	24	4	3	0	-1	0	1
	z_j	$40M$	$6M$	$7M$	$-M$	$-M$	M	M
	$z_j - c_j$		$6M - 6$	$7M - 3$	$-M$	$-M$	0	0

Once an artificial variable is selected as the leaving variable, it will never reenter the tableau, so it can be eliminated.

In Table A-17 the x_2 column was selected as the pivot column because $7M - 3$ is the largest positive value in the $z_j - c_j$ row. A_1 was selected as the leaving basic variable (and pivot row) because the quotient of 4 for this row was the minimum positive row value.

The second simplex tableau is developed using the simplex formulas presented earlier. It is shown in Table A-18. Notice that the A_1 column has been eliminated in the second simplex tableau. Once an artificial variable leaves the basic feasible solution, it will never return because of its high cost, M . Thus, like the booster rocket, it can be eliminated from the tableau. However, artificial variables are the *only* variables that can be treated this way.

Table A-18
The Second Simplex Tableau

c_j	Basic Variables	Quantity	6	3	0	0	M
			x_1	x_2	s_1	s_2	A_1
3	x_2	4	1/2	1	-1/4	0	0
M	A_2	12	5/2	0	3/4	-1	1
	z_j	$12M + 12$	$5M/2 + 3/2$	3	$-3/4 + 3/M4$	$-M$	M
	$z_j - c_j$		$5M/2 - 9/2$	0	$-3/4 + 3/M4$	$-M$	0

The third simplex tableau, with x_1 replacing A_2 , is shown in Table A-19. Both the A_1 and A_2 columns have been eliminated because both variables have left the solution. The x_1 row

is selected as the pivot row because it corresponds to the minimum positive ratio of 16. In selecting the pivot row, the -4 value for the x_2 row was not considered because the minimum *positive* value or zero is selected. Selecting the x_2 row would result in a negative quantity value for s_1 in the fourth tableau, which is not feasible.

Table A-19
The Third Simplex Tableau

			6	3	0	0
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2
3	x_2	8/5	0	1	-2/5	1/5
6	x_1	24/5	1	0	3/10	-2/5
	z_j	168/5	6	3	3/5	-9/5
	$z_j - c_j$		0	0	3/5	-9/5

The fourth simplex tableau, with s_1 replacing x_1 , is shown in Table A-20. Table A-20 is the optimal simplex tableau because the $z_j - c_j$ row contains no positive values. The optimal solution is

$$\begin{aligned}
 x_1 &= 0 \text{ bags of Super-gro} \\
 s_1 &= 16 \text{ extra lb of nitrogen} \\
 x_2 &= 8 \text{ bags of Crop-quick} \\
 s_2 &= 0 \text{ extra lb of phosphate} \\
 Z &= \$24, \text{ total cost of purchasing fertilizer}
 \end{aligned}$$

Table A-20
Optimal Simplex Tableau

			6	3	0	0
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2
3	x_2	8	4/3	1	0	-1/3
0	s_1	16	10/3	0	1	-4/3
	z_j	24	4	3	0	-1
	$z_j - c_j$		-2	0	0	-1

Simplex Adjustments for a Minimization Problem

To summarize, the adjustments necessary to apply the simplex method to a minimization problem are as follows:

1. Transform all \geq constraints to equations by subtracting a surplus variable and adding an artificial variable.
2. Assign a c_j value of M to each artificial variable in the objective function.
3. Change the $c_j - z_j$ row to $z_j - c_j$.

Although the fertilizer example model we just used included only \geq constraints, it is possible for a minimization problem to have \leq and $=$ constraints in addition to \geq constraints. Similarly, it is possible for a maximization problem to have \geq and $=$ constraints in addition to \leq constraints. Problems that contain a combination of different types of inequality constraints are referred to as *mixed constraint problems*.

A Mixed Constraint Problem

A mixed constraint problem includes a combination of \leq , $=$, and \geq constraints.

So far we have discussed maximization problems with all \leq constraints and minimization problems with all \geq constraints. However, we have yet to solve a problem with a mixture of \leq , \geq , and $=$ constraints. Furthermore, we have not yet looked at a maximization problem with a \geq constraint. The following is a maximization problem with \leq , \geq , and $=$ constraints.

A leather shop makes custom-designed, hand-tooled briefcases and luggage. The shop makes a \$400 profit from each briefcase and a \$200 profit from each piece of luggage. (The profit for briefcases is higher because briefcases require more hand tooling.) The shop has a contract to provide a store with exactly 30 items per month. A tannery supplies the shop with at least 80 square yards of leather per month. The shop must use at least this amount but can order more. Each briefcase requires 2 square yards of leather; each piece of luggage requires 8 square yards of leather. From past performance, the shop owners know they cannot make more than 20 briefcases per month. They want to know the number of briefcases and pieces of luggage to produce in order to maximize profit.

This problem is formulated as

$$\text{maximize } Z = \$400x_1 + 200x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 30 \text{ contracted items} \\ 2x_1 + 8x_2 &\geq 80 \text{ yd}^2 \text{ of leather} \\ x_1 &\leq 20 \text{ briefcases} \\ x_1, x_2 &\geq 0 \end{aligned}$$

where x_1 = briefcases and x_2 = pieces of luggage.

The first step in the simplex method is to transform the inequalities into equations. The first constraint for the contracted items is already an equation; therefore, it is not necessary to add a slack variable. There can be no slack in the contract with the store because exactly 30 items must be delivered. Even though this equation already appears to be in the necessary form for simplex solution, let us test it at the origin to see if it meets the starting requirements.

$$\begin{aligned} x_1 + x_2 &= 30 \\ 0 + 0 &= 30 \\ 0 &\neq 30 \end{aligned}$$

An artificial variable is added to an equality (=) constraint for standard form.

Because zero does not equal 30, the constraint is not feasible in this form. Recall that a \geq constraint did not work at the origin either in an earlier problem. Therefore, an artificial variable was added. The same thing can be done here.

$$x_1 + x_2 + A_1 = 30$$

Now at the origin, where $x_1 = 0$ and $x_2 = 0$,

$$\begin{aligned} 0 + 0 + A_1 &= 30 \\ A_1 &= 30 \end{aligned}$$

Any time a constraint is initially an equation, an artificial variable is added. However, the artificial variable cannot be assigned a value of M in the objective function of a maximization problem. Because the objective is to maximize profit, a positive M value would represent a large positive *profit* that would definitely end up in the final solution. Because an artificial variable has no real meaning and is inserted into the model merely to create an initial

solution at the origin, its existence in the final solution would render the solution meaningless. To prevent this from happening, we must give the artificial variable a large *cost* contribution, or $-M$.

An artificial variable in a maximization problem is given a large cost contribution to drive it out of the problem.

The constraint for leather is a \geq inequality. It is converted to equation form by subtracting a surplus variable and adding an artificial variable:

$$2x_1 + 8x_2 - s_1 + A_2 = 80$$

As in the equality constraint, the artificial variable in this constraint must be assigned an objective function coefficient of $-M$.

The final constraint is a \leq inequality and is transformed by adding a slack variable:

$$x_1 + s_2 = 20$$

The completely transformed linear programming problem is as follows:

$$\text{maximize } Z = 400x_1 + 200x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$$

subject to

$$\begin{aligned} x_1 + x_2 + A_1 &= 30 \\ 2x_1 + 8x_2 - s_1 + A_2 &= 80 \\ x_1 + s_2 &= 20 \\ x_1, x_2, s_1, s_2, A_1, A_2 &\geq 0 \end{aligned}$$

The initial simplex tableau for this model is shown in Table A-21. Notice that the basic solution variables are a mix of artificial and slack variables. Note also that the third-row quotient for determining the pivot row ($20 \div 0$) is an undefined value, or ∞ . Therefore, this row would never be considered as a candidate for the pivot row. The second, third, and optimal tableaus for this problem are shown in Tables A-22, A-23, and A-24.

Table A-21

The Initial Simplex Tableau

c_j	Basic Variables	Quantity	400	200	0	0	$-M$	$-M$
			x_1	x_2	s_1	s_2	A_1	A_2
$-M$	A_1	30	1	1	0	0	1	0
$-M$	A_2	80	2	8	-1	0	0	1
0	s_2	20	1	0	0	1	0	0
	z_j	$-110M$	$-3M$	$-9M$	M	0	$-M$	$-M$
	$c_j - z_j$		$400 + 3M$	$200 + 9M$	$-M$	0	0	0

Table A-22

The Second Simplex Tableau

c_j	Basic Variables	Quantity	400	200	0	0	$-M$
			x_1	x_2	s_1	s_2	A_1
$-M$	A_1	20	$3/4$	0	$1/8$	0	1
200	x_2	10	$1/4$	1	$-1/8$	0	0
0	s_2	20	1	0	0	1	0
	z_j	$2,000 - 20M$	$50 - 3M/4$	200	$-25 - M/8$	0	$-M$
	$c_j - z_j$		$350 + 3M/4$	0	$25 + M/8$	0	0

Table A-23
The Third Simplex Tableau

c_j	Basic Variables	Quantity	400	200	0	0	$-M$
			x_1	x_2	s_1	s_2	A_1
$-M$	A_1	5	0	0	1/8	$-3/4$	1
200	x_2	5	0	1	$-1/8$	$-1/4$	0
400	x_1	20	1	0	0	1	0
	z_j	$9,000 - 5M$	400	200	$-25 - M/8$	$350 + 3M/4$	$-M$
	$c_j - z_j$		0	0	$25 + M/8$	$-350 - 3M/4$	0

Table A-24
The Optimal Simplex Tableau

c_j	Basic Variables	Quantity	400	500	0	0
			x_1	x_2	s_1	s_2
0	s_1	40	0	0	1	-6
200	x_2	10	0	1	0	-1
400	x_1	20	1	0	0	1
	z_j	10,000	400	200	0	200
	$c_j - z_j$		0	0	0	-200

The solution for the leather shop problem is (see Table A-24):

- $x_1 = 20$ briefcases
- $x_2 = 10$ pieces of luggage
- $s_1 = 40$ extra yd^2 of leather
- $Z = \$10,000$ profit per month

It is now possible to summarize a set of rules for transforming all three types of model constraints.

Constraint	Adjustment	Objective Function Coefficient	
		MAXIMIZATION	MINIMIZATION
\leq	Add a slack variable	0	0
$=$	Add an artificial variable	$-M$	M
\geq	Subtract a surplus variable and add an artificial variable	$-M$	M

Irregular Types of Linear Programming Problems

For irregular problems the general simplex procedure does not always apply.

The basic simplex solution of typical maximization and minimization problems has been shown in this module. However, there are several special types of atypical linear programming problems. Although these special cases do not occur frequently, they will be described within the simplex framework so that you can recognize them when they arise.

These special types include problems with more than one optimal solution, infeasible problems, problems with unbounded solutions, problems with ties for the pivot column or ties for the pivot row, and problems with constraints with negative quantity values.

Multiple Optimal Solutions

Consider the Beaver Creek Pottery Company example with the objective function changed as follows.

$$Z = 40x_1 + 50x_2$$

to

$$\begin{aligned} Z &= 40x_1 + 30x_2 \\ \text{maximize } Z &= 40x_1 + 30x_2 \end{aligned}$$

subject to

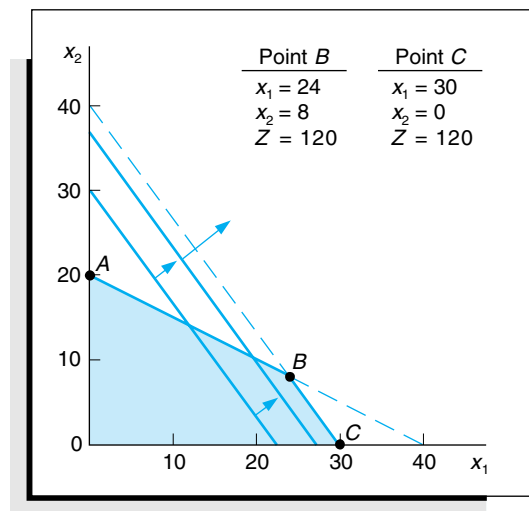
$$\begin{aligned} x_1 + 2x_2 &\leq 40 \\ 4x_1 + 3x_2 &\leq 120 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Alternate optimal solutions have the same Z value but different variable values.

The graph of this model is shown in Figure A-5. The slight change in the objective function makes it now *parallel* to the constraint line, $4x_1 + 3x_2 = 120$. Therefore, as the objective function edge moves outward from the origin, it touches the whole line segment *BC* rather than a single extreme corner point before it leaves the feasible solution area. The endpoints of this line segment, *B* and *C*, are typically referred to as the *alternate optimal solutions*. It is understood that these points represent the endpoints of a range of optimal solutions.

Figure A-5

Graph of the Beaver Creek Pottery Company example with multiple optimal solutions



For a multiple optimal solution the $c_j - z_j$ (or $z_j - c_j$) value for a nonbasic variable in the final tableau equals zero.

The optimal simplex tableau for this problem is shown in Table A-25. This corresponds to point *C* in Figure A-5.

The fact that this problem contains multiple optimal solutions can be determined from the $c_j - z_j$ row. Recall that the $c_j - z_j$ values are the net increases in profit per unit for the variable in each column. Thus, $c_j - z_j$ values of zero indicate no net increase in profit and no net loss in profit. We would expect the basic variables, s_1 and x_1 , to have zero $c_j - z_j$ values because they are part of the basic feasible solution; they are already in the solution so they cannot be entered again. However, the x_2 column has a $c_j - z_j$ value of zero and it is

not part of the basic feasible solution. This means that if some mugs (x_2) were produced, we would have a new product mix but the same total profit. Thus, a multiple optimal solution is indicated by a $c_j - z_j$ (or $z_j - c_j$) row value of zero for a nonbasic variable.

Table A-25

The Optimal Simplex Tableau

c_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
0	s_1	10	0	5/4	1	-1/4
40	x_1	30	1	3/4	0	1/4
	z_j	1,200	40	30	0	10
	$c_j - z_j$		0	0	0	-10

An alternate optimal solution is determined by selecting the non-basic variable with $c_j - z_j = 0$ as the entering variable.

To determine the alternate endpoint solution, let x_2 be the entering variable (pivot column) and select the pivot row as usual. This selection results in the s_1 row being the pivot row. The alternate solution corresponding to point B in Figure A-5 is shown in Table A-26.

Table A-26

The Alternative Optimal Tableau

c_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
40	x_2	8	0	1	4/5	-1/5
30	x_1	24	1	0	-3/5	2/5
	z_j	1,200	40	30	0	10
	$c_j - z_j$		0	0	0	-10

An Infeasible Problem

An infeasible problem does not have a feasible solution space.

Another linear programming irregularity is the case where a problem has no feasible solution area; thus, there is no basic feasible solution to the problem.

An example of an infeasible problem is formulated next and depicted graphically in Figure A-6.

$$\text{maximize } Z = 5x_1 + 3x_2$$

subject to

$$4x_1 + 2x_2 \leq 8$$

$$x_1 \geq 4$$

$$x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

The three constraints do not overlap to form a feasible solution area. Because no point satisfies all three constraints simultaneously, there is no solution to the problem. The final simplex tableau for this problem is shown in Table A-27.

The tableau in Table A-27 has all zero or negative values in the $c_j - z_j$ row, indicating that it is optimal. However, the solution is $x_2 = 4, A_1 = 4,$ and $A_2 = 2$. Because the existence of artificial variables in the final solution makes the solution meaningless, this is not a real

An infeasible problem has an artificial variable in the final simplex tableau.

Figure A-6

Graph of an infeasible problem

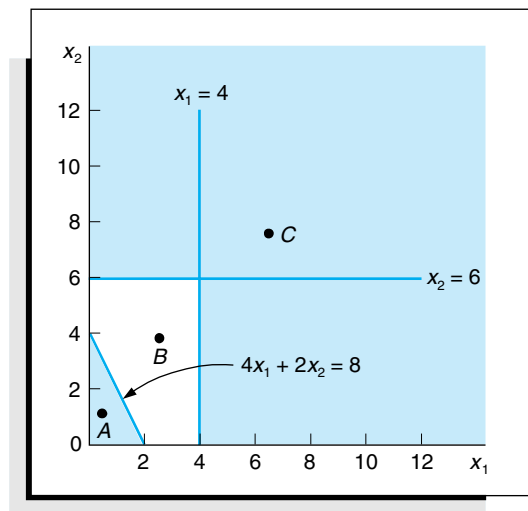


Table A-27

The Final Simplex Tableau for an Infeasible Problem

c_j	Basic Variables	Quantity	5	3	0	0	0	-M	-M
			x_1	x_2	s_1	s_2	s_3	A_1	A_2
3	x_2	4	2	1	1/2	0	0	0	0
-M	A_1	4	1	0	0	-1	0	1	0
-M	A_2	2	-2	0	-1/2	0	-1	0	1
	z_j	$12 - 6M$	$6 + M$	3	$3/2 + M/2$	M	M	-M	-M
	$c_j - z_j$		$-1 - M$	0	$-3/2 - M/2$	-M	-M	0	0

solution. In general, any time the $c_j - z_j$ (or $z_j - c_j$) row indicates that the solution is optimal but there are artificial variables in the solution, the solution is infeasible. Infeasible problems do not typically occur, but when they do they are usually a result of errors in defining the problem or in formulating the linear programming model.

An Unbounded Problem

In an unbounded problem the objective function can increase indefinitely because the solution space is not closed.

In some problems the feasible solution area formed by the model constraints is not closed. In these cases it is possible for the objective function to increase indefinitely without ever reaching a maximum value because it never reaches the boundary of the feasible solution area.

An example of this type of problem is formulated next and shown graphically in Figure A-7.

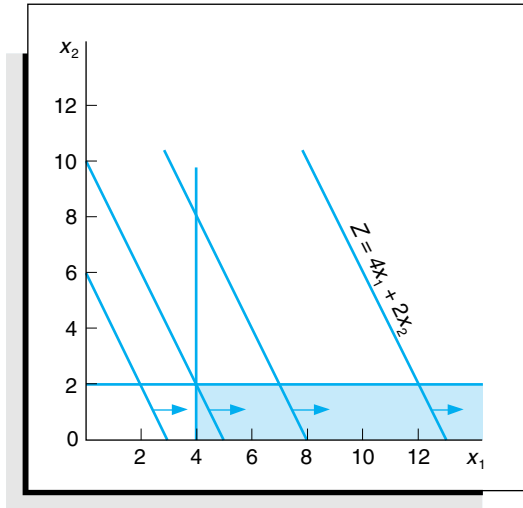
$$\text{maximize } Z = 4x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 &\geq 4 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Figure A-7

An unbounded problem



In Figure A-7 the objective function is shown to increase without bound; thus, a solution is never reached.

A pivot row cannot be selected for an unbounded problem.

The second tableau for this problem is shown in Table A-28. In this simplex tableau, s_1 is chosen as the entering nonbasic variable and pivot column. However, there is no pivot row or leaving basic variable. One row value is -4 and the other is undefined. This indicates that a “most constrained” point does not exist and that the solution is unbounded. In general, a solution is unbounded if the row value ratios are all negative or undefined.

Table A-28

The Second Simplex Tableau

c_j	Basic Variables	Quantity	4	2	0	0	
			x_1	x_2	s_1	s_2	
4	x_1	4	0	0	-1	0	$4 \div -1 = -4$
0	s_2	2	1	1	0	1	$2 \div 0 = \infty$
	z_j	16	4	0	-4	0	
	$c_j - z_j$		0	2	4	0	

Unlimited profits are not possible in the real world; an unbounded solution, like an infeasible solution, typically reflects an error in defining the problem or in formulating the model.

Tie for the Pivot Column

A tie for the pivot column is broken arbitrarily.

Sometimes when selecting the pivot column, you may notice that the greatest positive $c_j - z_j$ (or $z_j - c_j$) row values are the same; thus, there is a tie for the pivot column. When this happens, one of the two tied columns should be selected arbitrarily. Even though one choice may require fewer subsequent iterations than the other, there is no way of knowing this beforehand.

Tie for the Pivot Row—Degeneracy

It is also possible to have a tie for the pivot row (i.e., two rows may have identical lowest nonnegative values). Like a tie for a pivot column, a tie for a pivot row should be broken arbitrarily. However, after the tie is broken, the basic variable that was the *other* choice for

A tie for the pivot row is broken arbitrarily and can lead to degeneracy.

the leaving basic variable will have a quantity value of zero in the next tableau. This condition is commonly referred to as *degeneracy* because theoretically it is possible for subsequent simplex tableau solutions to degenerate so that the objective function value never improves and optimality never results. This occurs infrequently, however.

In general, tableaus with ties for the pivot row should be treated normally. If the simplex steps are carried out as usual, the solution will evolve normally.

The following is an example of a problem containing a tie for the pivot row.

$$\text{maximize } Z = 4x_1 + 6x_2$$

subject to

$$6x_1 + 4x_2 \leq 24$$

$$x_2 \leq 3$$

$$5x_1 + 10x_2 \leq 40$$

$$x_1, x_2 \geq 0$$

For the sake of brevity we will skip the initial simplex tableau for this problem and go directly to the second simplex tableau in Table A-29, which shows a tie for the pivot row between the s_1 and s_3 rows.

Table A-29
The Second Simplex Tableau with a Tie for the Pivot Row

			4	6	0	0	0		
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2	s_3		
0	s_1	12	6	0	1	-4	0	$12 \div 6 = 2$	} Tie
6	x_2	3	0	1	0	1	0		
0	s_3	10	5	0	0	-10	1	$10 \div 5 = 2$	
	z_j	18	0	6	0	6	0		
	$c_j - z_j$		4	0	0	-6	0		

The s_3 row is selected arbitrarily as the pivot row, resulting in the third simplex tableau, shown in Table A-30.

Table A-30
The Third Simplex Tableau with Degeneracy

			4	6	0	0	0		
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2	s_3		
0	s_1	0	0	0	1	8	-6/5		
6	x_2	3	0	1	0	1	0		
4	x_1	2	1	0	0	-2	1/5		
	z_j	26	4	6	0	-2	4/5		
	$c_j - z_j$		0	0	0	2	-4/5		

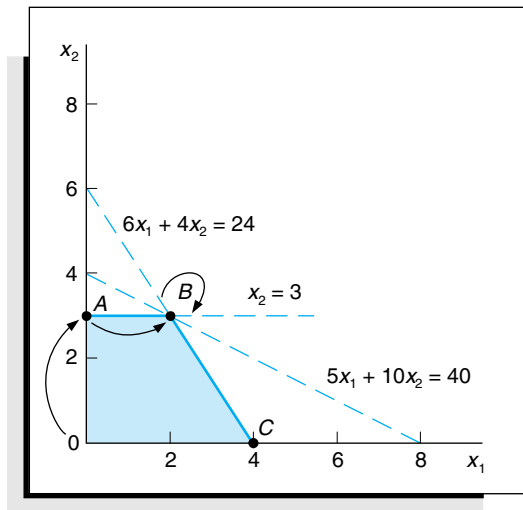
Note that in Table A-30 a quantity value of zero now appears in the s_1 row, representing the degenerate condition resulting from the tie for the pivot row. However, the simplex process should be continued as usual: s_2 should be selected as the entering basic variable and the s_1 row should be selected as the pivot row. (Recall that the pivot row value of zero is the minimum nonnegative quotient.) The final optimal tableau is shown in Table A-31.

Table A-31
The Optimal Simplex Tableau for a Degenerate Problem

c_j	Basic Variables	Quantity					
			4	6	0	0	0
			x_1	x_2	s_1	s_2	s_3
0	s_2	0	0	0	1/8	1	-3/20
6	x_2	3	0	1	-1/8	0	3/20
4	x_1	2	1	0	1/4	0	-1/10
	z_j	26	4	6	1/4	0	1/2
	$c_j - z_j$		0	0	-1/4	0	-1/2

Notice that the optimal solution did not change from the third to the optimal simplex tableau. The graphical analysis of this problem shown in Figure A-8 reveals the reason for this.

Figure A-8
Graph of a degenerate solution



Degeneracy occurs in a simplex problem when a problem continually loops back to the same solution or tableau.

Notice that in the third tableau (Table A-30) the simplex process went to point B, where all three constraint lines intersect. This is, in fact, what caused the tie for the pivot row and the degeneracy. Subsequently, the simplex process stayed at point B in the optimal tableau (Table A-31). The two tableaus represent two different basic feasible solutions corresponding to two different sets of model constraint equations.

Negative Quantity Values

Standard form for simplex solution requires positive right-hand-side values.

A negative right-hand-side value is changed to a positive by multiplying the constraint by -1 , which changes the inequality sign.

Occasionally a model constraint is formulated with a negative quantity value on the right side of the inequality sign—for example,

$$-6x_1 + 2x_2 \geq -30$$

This is an improper condition for the simplex method, because for the method to work, all quantity values must be positive or zero.

This difficulty can be alleviated by multiplying the inequality by -1 , which also changes the direction of the inequality.

$$(-1)(-6x_1 + 2x_2 \geq -30) \\ 6x_1 - 2x_2 \leq 30$$

Now the model constraint is in proper form to be transformed into an equation and solved by the simplex method.

Summary of Simplex Irregularities

Multiple optimal solutions are identified by $c_j - z_j$ (or $z_j - c_j$) = 0 for a nonbasic variable. To determine the alternate solution(s), enter the nonbasic variable(s) with a $c_j - z_j$ value equal to zero.

An infeasible problem is identified in the simplex procedure when an optimal solution is achieved (i.e., when all $c_j - z_j \leq 0$) and one or more of the basic variables are artificial.

An unbounded problem is identified in the simplex procedure when it is not possible to select a pivot row—that is, when the values obtained by dividing the quantity values by the corresponding pivot column values are negative or undefined.

The Dual

*The original linear programming model is called the **primal**, and the alternative form is the **dual**.*

Every linear programming model has two forms: the *primal* and the *dual*. The original form of a linear programming model is called the primal. All the examples in this module are primal models. The dual is an alternative model form derived completely from the primal. The dual is useful because it provides the decision maker with an alternative way of looking at a problem. Whereas the primal gives solution results in terms of the amount of profit gained from producing products, the dual provides information on the *value* of the constrained resources in achieving that profit.

The dual solution variables provide the value of the resources, that is, shadow prices.

The following example will demonstrate how the dual form of a model is derived and what it means. The Hickory Furniture Company produces tables and chairs on a daily basis. Each table produced results in \$160 in profit; each chair results in \$200 in profit. The production of tables and chairs is dependent on the availability of limited resources—labor, wood, and storage space. The resource requirements for the production of tables and chairs and the total resources available are as follows.

Resource	Resource Requirements		Total Available per Day
	TABLE	CHAIR	
Labor (hr)	2	4	40
Wood (bd ft)	18	18	216
Storage (ft ²)	24	12	240

The company wants to know the number of tables and chairs to produce per day to maximize profit. The model for this problem is formulated as follows.

$$\text{maximize } Z = \$160x_1 + 200x_2$$

subject to

$$\begin{aligned}
 2x_1 + 4x_2 &\leq 40 \text{ hr of labor} \\
 18x_1 + 18x_2 &\leq 216 \text{ bd ft of wood} \\
 24x_1 + 12x_2 &\leq 240 \text{ ft}^2 \text{ of storage space} \\
 x_1, x_2 &\geq 0
 \end{aligned}$$

where

$$\begin{aligned}x_1 &= \text{number of tables produced} \\x_2 &= \text{number of chairs produced}\end{aligned}$$

This model represents the primal form. For a primal maximization model, the dual form is a minimization model. The dual form of this example model is

$$\text{minimize } Z = 40y_1 + 216y_2 + 240y_3$$

subject to

$$\begin{aligned}2y_1 + 18y_2 + 24y_3 &\geq 160 \\4y_1 + 18y_2 + 12y_3 &\geq 200 \\y_1, y_2, y_3 &\geq 0\end{aligned}$$

The dual is formulated entirely from the primal.

The specific relationships between the primal and the dual demonstrated in this example are as follows.

1. The dual variables, y_1 , y_2 , and y_3 , correspond to the model constraints in the primal. For every constraint in the primal there will be a variable in the dual. For example, in this case the primal has *three constraints*; therefore, the dual has *three decision variables*.
2. The quantity values on the right-hand side of the primal inequality constraints are the objective function coefficients in the dual. The constraint quantity values in the primal, 40, 216, and 240, form the dual objective function: $Z = 40y_1 + 216y_2 + 240y_3$.
3. The model constraint coefficients in the primal are the decision variable coefficients in the dual. For example, the labor constraint in the primal has the coefficients 2 and 4. These values are the y_1 variable coefficients in the model constraints of the dual: $2y_1$ and $4y_1$.
4. The objective function coefficients in the primal, 160 and 200, represent the model constraint requirements (quantity values on the right-hand side of the constraint) in the dual.
5. Whereas the maximization primal model has \leq constraints, the minimization dual model has \geq constraints.

The primal–dual relationships can be observed by comparing the two model forms shown in Figure A-9.

A primal maximization model with \leq constraints converts to a dual minimization model with \geq constraints, and vice versa.

Now that we have developed the dual form of the model, the next step is determining what the dual means. In other words, what do the decision variables y_1 , y_2 , and y_3 mean, what do the \geq model constraints mean, and what is being minimized in the dual objective function?

Interpreting the Dual Model

The dual model can be interpreted by observing the simplex solution to the primal form of the model. The simplex solution to the primal model is shown in Table A-32.

Interpreting this primal solution, we have

$$\begin{aligned}x_1 &= 4 \text{ tables} \\x_2 &= 8 \text{ chairs} \\s_3 &= 48 \text{ ft}^2 \text{ of storage space} \\Z &= \$2,240 \text{ profit}\end{aligned}$$

Figure A-9

The primal–dual relationships

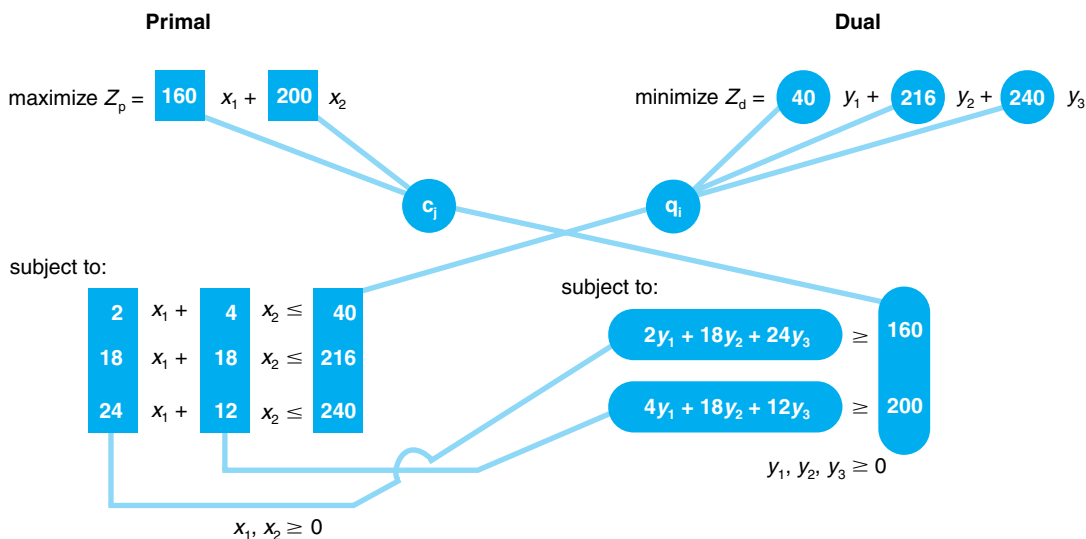


Table A-32

The Optimal Simplex Solution for the Primal Model

c_j	Basic Variables	Quantity	160	200	0	0	0
			x_1	x_2	s_1	s_2	s_3
200	x_2	8	0	1	1/2	-1/18	0
160	x_1	4	1	0	-1/2	1/9	0
0	s_3	48	0	0	6	-2	1
	z_j	2,240	160	200	20	20/3	0
	$c_j - z_j$		0	0	-20	-20/3	0

This optimal primal tableau also contains information about the dual. In the $c_j - z_j$ row of Table A-32, the negative values of -20 and $-20/3$ under the s_1 and s_2 columns indicate that if one unit of either s_1 or s_2 were entered into the solution, profit would *decrease* by \$20 or \$6.67 (i.e., $20/3$), respectively.

Recall that s_1 represents unused labor and s_2 represents unused wood. In the present solution s_1 and s_2 are not basic variables, so they both equal zero. This means that all of the material and labor are being used to make tables and chairs, and there are no excess (slack) labor hours or board feet of material left over. Thus, if we enter s_1 or s_2 into the solution, then s_1 or s_2 no longer equals zero, we would be decreasing the use of labor or wood. If, for example, one unit of s_1 is entered into the solution, then one unit of labor previously used is not used, and profit is reduced by \$20.

Let us assume that one unit of s_1 has been entered into the solution so that we have one hour of unused labor ($s_1 = 1$). Now let us remove this unused hour of labor from the solution so that all labor is again being used. We previously noted that profit was decreased by \$20 by entering one hour of unused labor; thus, it can be expected that if we take this hour back (and use it again), profit will be increased by \$20. This is analogous to saying that if we could get one more hour of labor, we could increase profit by \$20. Therefore, if we could purchase one hour of labor, we would be willing to pay up to \$20 for it because that is the amount by which it would increase profit.

Time Out

for John Von Neumann

John Von Neumann, the famous Hungarian mathematician, is credited with many contributions in science and mathematics, including crucial work on the development of the atomic bomb during World War II and the development of the computer following the war. In 1947 George Dantzig visited Von

Neumann at the Institute for Advanced Study at Princeton and described the linear programming technique to him. Von Neumann grasped the technique immediately, because he had been working on similar concepts himself, and went on to explain duality to Dantzig for the first time.

The $c_j - z_j$ values for slack variables are the marginal values of the constraint resources, i.e., shadow prices.

If a resource is not completely used, i.e., there is slack, its marginal value is zero.

The **shadow price** is the **maximum** amount that should be paid for one additional unit of a resource.

The total marginal value of the resources equals the optimal profit.

The negative $c_j - z_j$ row values of \$20 and \$6.67 are the *marginal values* of labor (s_1) and wood (s_2), respectively. These dual values are also often referred to as *shadow prices*, since they reflect the maximum “price” one would be willing to pay to obtain one more unit of the resource.

What happened to the third resource, storage space? The answer can be seen in Table A-32. Notice that the $c_j - z_j$ row value for s_3 (which represents unused storage space) is zero. This means that storage space has a marginal value of zero; that is, we would not be willing to pay anything for an extra foot of storage space.

The reason more storage space has no marginal value is because storage space was not a limitation in the production of tables and chairs. Table A-32 shows that 48 square feet of storage space were left unused (i.e., $s_3 = 48$) after the 4 tables and 8 chairs were produced. Since the company already has 48 square feet of storage space left over, an extra square foot would have no additional value; the company cannot even use all of the storage space it has available.

We need to consider one additional aspect of these marginal values. In our discussion of the marginal value of these resources, we have indicated that the marginal value (or *shadow price*) is the *maximum* amount that would be paid for additional resources. The marginal value of \$60 for one hour of labor is not necessarily what the Hickory Furniture Company would *pay* for an hour of labor. This depends on how the objective function is defined. In this example we are assuming that all of the resources available, 40 hours of labor, 216 board feet of wood, and 240 square feet of storage space, are already paid for. Even if the company does not use all the resources, it still must pay for them. They are *sunk* costs. In other words, the cost of any additional resources secured are included in the objective function coefficients. As such, the profit values in the objective function for each product are unaffected by how much of a resource is actually used; the profit is independent of the resources used. If the cost of the resources is not included in the profit function, then securing additional resources would reduce the marginal value.

Continuing our analysis, we note that the profit in the primal model was shown to be \$2,240. For the furniture company, the value of the resources used to produce tables and chairs must be in terms of this profit. In other words, the value of the labor and wood resources is determined by their contribution toward the \$2,240 profit. Thus, if the company wanted to assign a *value* to the resources it used, it could not assign an amount greater than the profit earned by the resources. Conversely, using the same logic, the total *value* of the resources must also be at least as much as the profit they earn. Thus, the value of all the resources must *exactly equal* the profit earned by the optimal solution.

Now let us look again at the dual form of the model.

$$\text{minimize } Z_d = 40y_1 + 16y_2 + 240y_3$$

subject to

$$2y_1 + 18y_2 + 24y_3 \geq 160$$

$$4y_1 + 18y_2 + 12y_3 \geq 200$$

$$y_1, y_2, y_3 \geq 0$$

The dual variables equal the marginal value of the resources, the shadow prices.

Given the previous discussion on the value of the model resources, we can now define the decision variables of the dual, y_1 , y_2 , and y_3 , to represent the marginal values of the resources:

$$y_1 = \text{marginal value of 1 hr of labor} = \$20$$

$$y_2 = \text{marginal value of 1 bd ft of wood} = \$6.67$$

$$y_3 = \text{marginal value of 1 ft}^2 \text{ of storage space} = \$0$$

Use of the Dual

The dual provides the decision maker with a basis for deciding how much to pay for more resources.

The importance of the dual to the decision maker lies in the information it provides about the model resources. Often the manager is less concerned about profit than about the use of resources because the manager often has more control over the use of resources than over the accumulation of profits. The dual solution informs the manager of the value of the resources, which is important in deciding whether or not to secure more resources and how much to pay for these additional resources.

If the manager secures more resources, the next question is “How does this affect the original solution?” The feasible solution area is determined by the values forming the model constraints, and if those values are changed, it is possible for the feasible solution area to change. The effect on the solution of changes to the model is the subject of sensitivity analysis, the next topic to be presented here.

Sensitivity Analysis

In this section we will show how sensitivity ranges are mathematically determined using the simplex method. While this is not as efficient or quick as using the computer, close examination of the simplex method for performing sensitivity analysis can provide a more thorough understanding of the topic.

Changes in Objective Function Coefficients

To demonstrate sensitivity analysis for the coefficients in the objective function, we will use the Hickory Furniture Company example developed in the previous section. The model for this example was formulated as

$$\text{maximize } Z = \$160x_1 + 200x_2$$

subject to

$$2x_1 + 4x_2 \leq 40 \text{ hr of labor}$$

$$18x_1 + 18x_2 \leq 216 \text{ bd ft of wood}$$

$$24x_1 + 12x_2 \leq 240 \text{ ft}^2 \text{ of storage space}$$

$$x_1, x_2 \geq 0$$

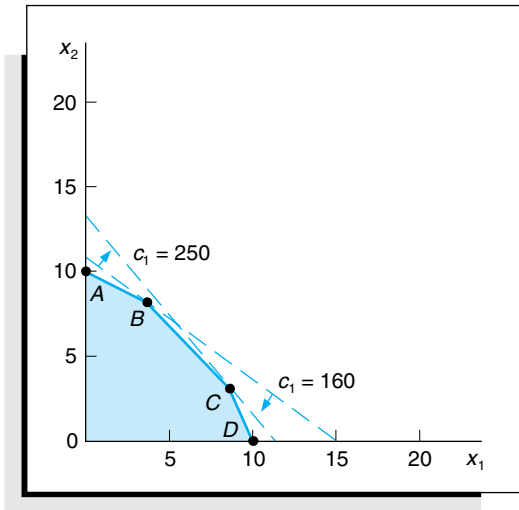
where

$$x_1 = \text{number of tables produced}$$

$$x_2 = \text{number of chairs produced}$$

The coefficients in the objective function will be represented symbolically as c_j (the same notation used in the simplex tableau). Thus, $c_1 = 160$ and $c_2 = 200$. Now, let us consider a change in one of the c_j values by an amount Δ . For example, let us change $c_1 = 160$ by $\Delta = 90$. In other words, we are changing c_1 from \$160 to \$250. The effect of this change on the solution of this model is shown graphically in Figure A-10.

Figure A-10
A change in c_1



The sensitivity range for a c_j value is the range of values over which the current optimal solution will remain optimal.

Originally, the solution to this problem was located at point B in Figure A-10, where $x_1 = 4$ and $x_2 = 8$. However, increasing c_1 from \$160 to \$250 shifts the slope of the objective function so that point C ($x_1 = 8, x_2 = 4$) becomes the optimal solution. This demonstrates that a change in one of the coefficients of the objective function can change the optimal solution. Therefore, sensitivity analysis is performed to determine the range over which c_j can be changed without altering the optimal solution.

The range of c_j that will maintain the optimal solution can be determined directly from the optimal simplex tableau. The optimal simplex tableau for our furniture company example is shown in Table A-33.

Table A-33
The Optimal Simplex Tableau

c_j	Basic Variables	Quantity	160	200	0	0	0
			x_1	x_2	s_1	s_2	s_3
200	x_2	8	0	1	1/2	-1/18	0
160	x_1	4	1	0	-1/2	1/9	0
0	s_3	48	0	0	6	-2	1
	z_j	2,240	160	200	20	20/3	0
	$c_j - z_j$		0	0	-20	-20/3	0

Δ is added to c_j in the optimal simplex tableau.

First, consider a Δ change for c_1 . This will change the c_1 value from $c_1 = 160$ to $c_1 = 160 + \Delta$, as shown in Table A-34. Notice that when c_1 is changed to $160 + \Delta$, the new value is included not only in the top c_j row but also in the left-hand c_j column. This is because x_1 is a basic solution variable. Since $160 + \Delta$ is in the left-hand column, it becomes

a multiple of the column values when the new z_j row values and the subsequent $c_j - z_j$ row values, also shown in Table A-34, are computed.

Table A-34
The Optimal Simplex Tableau
with $c_1 = 160 + \Delta$

c_j	Basic Variables	Quantity	$160 + \Delta$	200	0	0	0
			x_1	x_2	s_1	s_2	s_3
200	x_2	8	0	1	1/2	-1/18	0
$160 + \Delta$	x_1	4	1	0	-1/2	1/9	0
0	s_3	48	0	0	6	-2	1
	z_j	$2,240 + 4\Delta$	$160 + \Delta$	200	$20 - \Delta/2$	$20/3 + \Delta/9$	0
	$c_j - z_j$		0	0	$-20 + \Delta/2$	$-20/3 - \Delta/9$	0

For the solution to remain optimal all values in the $c_j - z_j$ row must be ≤ 0 .

The solution shown in Table A-34 will remain optimal as long as the $c_j - z_j$ row values remain negative. (If $c_j - z_j$ becomes positive, the product mix will change, and if it becomes zero, there will be an alternative solution.) Thus, for the solution to remain optimal,

$$-20 + \Delta/2 \leq 0$$

and

$$-20/3 - \Delta/9 \leq 0$$

Both of these inequalities must be solved for Δ .

$$\begin{aligned} -20 + \Delta/2 &\leq 0 \\ \Delta/2 &\leq 20 \\ \Delta &\leq 40 \end{aligned}$$

and

$$\begin{aligned} -20/3 - \Delta/9 &\leq 0 \\ -\Delta/9 &\leq 20/3 \\ -\Delta &\leq 60 \\ \Delta &\geq -60 \end{aligned}$$

Thus, $\Delta \leq 40$ and $\Delta \geq -60$. Now recall that $c_1 = 160 + \Delta$; therefore, $\Delta = c_1 - 160$. Substituting the amount $c_1 - 160$ for Δ in these inequalities,

$$\begin{aligned} \Delta &\leq 40 \\ c_1 - 160 &\leq 40 \\ c_1 &\leq 200 \end{aligned}$$

and

$$\begin{aligned} \Delta &\geq -60 \\ c_1 - 160 &\geq -60 \\ c_1 &\geq 100 \end{aligned}$$

Therefore, the range of values for c_1 over which the solution basis will remain optimal (although the value of the objective function may change) is

$$100 \leq c_1 \leq 200$$

Next, consider a Δ change in c_2 so that $c_2 = 200 + \Delta$. The effect of this change in the final simplex tableau is shown in Table A-35.

Table A-35
The Optimal Simplex Tableau
with $c_2 = 200 + \Delta$

c_j	Basic Variables	Quantity	160	$200 + \Delta$	0	0	0
			x_1	x_2	s_1	s_2	s_3
$200 + \Delta$	x_2	8	0	1	1/2	-1/18	0
160	x_1	4	1	0	-1/2	1/9	0
0	s_3	48	0	0	6	-2	1
	z_j	$2,240 + 8\Delta$	160	$200 + \Delta$	$20 + \Delta/2$	$20/3 - \Delta/18$	0
	$c_j - z_j$		0	0	$-20 - \Delta/2$	$-20/3 + \Delta/18$	0

As before, the solution shown in Table A-35 will remain optimal as long as the $c_j - z_j$ row values remain negative or zero. Thus, for the solution to remain optimal, we must have

$$-20 - \Delta/2 \leq 0$$

and

$$-20/3 + \Delta/18 \leq 0$$

Solving these inequalities for Δ gives

$$\begin{aligned} -20 - \Delta/2 &\leq 0 \\ -\Delta/2 &\leq 20 \\ \Delta &\geq -40 \end{aligned}$$

and

$$\begin{aligned} -20/3 + \Delta/18 &\leq 0 \\ \Delta/18 &\leq 20/3 \\ \Delta &\leq 120 \end{aligned}$$

Thus, $\Delta \geq -40$ and $\Delta \leq 120$. Since $c_2 = 200 + \Delta$, we have $\Delta = c_2 - 200$. Substituting this value for Δ in the inequalities yields

$$\begin{aligned} \Delta &\geq -40 \\ c_2 - 200 &\geq -40 \\ c_2 &\geq 160 \end{aligned}$$

and

$$\begin{aligned} \Delta &\leq 120 \\ c_2 - 200 &\leq 120 \\ c_2 &\leq 320 \end{aligned}$$

Therefore, the range of values for c_2 over which the solution will remain optimal is

$$160 \leq c_2 \leq 320$$

The ranges for both objective function coefficients are as follows.

$$\begin{aligned} 100 &\leq c_1 \leq 200 \\ 160 &\leq c_2 \leq 320 \end{aligned}$$

However, these ranges reflect a possible change in either c_1 or c_2 , not simultaneous changes in both c_1 and c_2 . Both of the objective function coefficients in this example were for basic solution variables. Determining the c_j sensitivity range for a decision variable that is *not basic* is much simpler. Because it is not in the basic variable column, the Δ change

does not become a multiple of the z_j row. Thus, the Δ change will show up in only one column in the $c_j - z_j$ row.

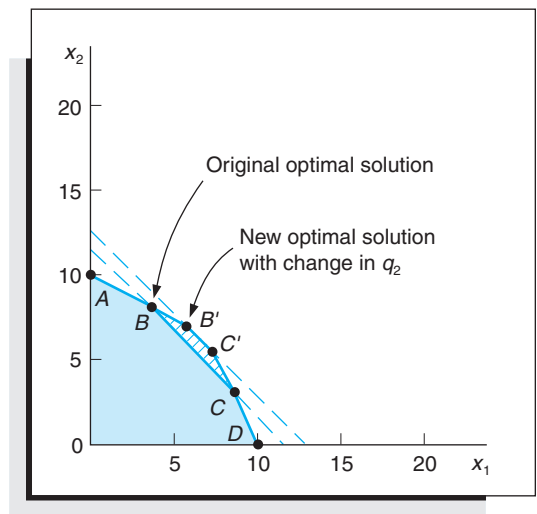
Changes in Constraint Quantity Values

To demonstrate the effect of a change in the quantity values of the model constraints, we will again use the Hickory Furniture Company example.

$$\begin{aligned} &\text{maximize } Z = 160x_1 + 200x_2 \\ &\text{subject to} \\ &\quad 2x_1 + 4x_2 \leq 40 \text{ hr of labor} \\ &\quad 18x_1 + 18x_2 \leq 216 \text{ bd ft of wood} \\ &\quad 24x_1 + 12x_2 \leq 240 \text{ ft}^2 \text{ of storage space} \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

The quantity values 40, 216, and 240 will be represented symbolically as q_i . Thus, $q_1 = 40$, $q_2 = 216$, and $q_3 = 240$. Now consider a Δ change in q_2 . For example, let us change $q_2 = 216$ by $\Delta = 18$. In other words, q_2 is changed from 216 board feet to 234 board feet. The effect of this change is shown graphically in Figure A-11.

Figure A-11
A Δ change in q_2



In Figure A-11 a change in q_2 is shown to have the effect of changing the feasible solution area from $OABCD$ to $OAB'C'D$. Originally, the optimal solution point was B ; however, the change in q_2 causes B' to be the new optimal solution point. At point B the optimal solution is

$$\begin{aligned} x_1 &= 4 \\ x_2 &= 8 \\ s_3 &= 48 \\ s_1 \text{ and } s_2 &= 0 \\ Z &= \$2,240 \end{aligned}$$

At point B' the new optimal solution is

$$\begin{aligned} x_1 &= 6 \\ x_2 &= 7 \end{aligned}$$

$$\begin{aligned} s_3 &= 12 \\ s_1 \text{ and } s_2 &= 0 \\ Z &= \$2,360 \end{aligned}$$

Thus, a change in a q_i value can change the values of the optimal solution. At some point an increase or decrease in q_i will change the variables in the optimal solution mix, including the slack variables. For example, if q_2 increases to 240 board feet, then the optimal solution point will be at

$$\begin{aligned} x_1 &= 6.67 \\ x_2 &= 6.67 \\ s_3 &= 0 \\ s_1 \text{ and } s_2 &= 0 \\ Z &= \$2,400 \end{aligned}$$

The sensitivity range for a q_i value is the range of values over which the right-hand-side values can vary without changing the solution variable mix, including slack variables and the shadow prices.

Notice that s_3 has left the solution; thus, the optimal solution mix has changed. At this point, where $q_2 = 240$, which is the upper limit of the sensitivity range for q_2 , the shadow price will also change. Therefore, the purpose of sensitivity analysis is to determine the range for q_i over which the optimal variable mix will remain the same and the shadow price will remain the same.

As in the case of the c_j values, the range for q_i can be determined directly from the optimal simplex tableau. As an example, consider a Δ increase in the number of labor hours. The model constraints become

$$\begin{aligned} 2x_1 + 4x_2 &\leq 40 + 1\Delta \\ 18x_1 + 18x_2 &\leq 216 + 0\Delta \\ 24x_1 + 12x_2 &\leq 240 + 0\Delta \end{aligned}$$

Notice in the initial simplex tableau for our example (Table A-36) that the changes in the quantity column are the same as the coefficients in the s_1 column.

Table A-36
The Initial Simplex Tableau

c_j	Basic Variables	Quantity	160	200	0	0	0
			x_1	x_2	s_1	s_2	s_3
0	s_1	$40 + 1\Delta$	2	4	1	0	0
0	s_2	$216 + 0\Delta$	18	18	0	1	0
0	s_3	$240 + 0\Delta$	24	12	0	0	1
	z_j	0	0	0	0	0	0
	$c_j - z_j$		160	200	0	0	0

A Δ in a q_i value is duplicated in the s_1 column in the final tableau.

This duplication will carry through each subsequent tableau, so the s_1 column values will duplicate the Δ changes in the quantity column in the final tableau (Table A-37).

In effect, the Δ changes form a separate column identical to the s_1 column. Therefore, to determine the Δ change, we need only observe the slack (s_i) column corresponding to the model constraint quantity (q_i) being changed.

Recall that a requirement of the simplex method is that the quantity values not be negative. If any q_i value becomes negative, the current solution will no longer be *feasible* and a new variable will enter the solution. Thus, the inequalities

Table A-37
The Final Simplex Tableau

c_j	Basic Variables	Quantity	160	200	0	0	0
			x_1	x_2	s_1	s_2	s_3
200	x_2	$8 + \Delta/2$	0	1	$1/2$	$-1/18$	0
160	x_1	$4 - \Delta/2$	1	0	$-1/2$	$1/9$	0
0	s_3	$48 + 6\Delta$	0	0	6	-2	1
	z_j	$2,240 + 20\Delta$	160	200	20	$20/3$	0
	$c_j - z_j$		0	0	-20	$-20/3$	0

$$8 + \Delta/2 \geq 0$$

$$4 - \Delta/2 \geq 0$$

$$48 + 6\Delta \geq 0$$

are solved for Δ :

$$8 + \Delta/2 \geq 0$$

$$\Delta/2 \geq -8$$

$$\Delta \geq -16$$

and

$$4 - \Delta/2 \geq 0$$

$$-\Delta/2 \geq -4$$

$$\Delta \leq 8$$

and

$$48 + 6\Delta \geq 0$$

$$6\Delta \geq -48$$

$$\Delta \geq -8$$

Since

$$q_1 = 40 + \Delta$$

then

$$\Delta = q_1 - 40$$

These values are substituted into the inequalities $\Delta \geq -16$, $\Delta \leq 8$, and $\Delta \geq -8$ as follows.

$$\Delta \geq -16$$

$$q_1 - 40 \geq -16$$

$$q_1 \geq 24$$

$$\Delta \leq 8$$

$$q_1 - 40 \leq 8$$

$$q_1 \leq 48$$

$$\Delta \geq -8$$

$$q_1 - 40 \geq -8$$

$$q_1 \geq 32$$

Summarizing these inequalities, we have

$$24 \leq 32 \leq q_1 \leq 48$$

The value of 24 can be eliminated, since q_1 must be greater than 32; thus,

$$32 \leq q_1 \leq 48$$

As long as q_1 remains in this range, the present basic solution variables will remain the same and feasible. However, the quantity values of those basic variables may change. In other words, although the variables in the basis remain the same, their values can change.

To determine the range for q_2 (where $q_2 = 216 + \Delta$), the s_2 column values are used to develop the Δ inequalities.

$$\begin{aligned} 8 - \Delta/18 &\geq 0 \\ 4 + \Delta/9 &\geq 0 \\ 48 - 2\Delta &\geq 0 \end{aligned}$$

The inequalities are solved as follows.

$$\begin{aligned} 8 - \Delta/18 &\geq 0 \\ -\Delta/18 &\geq -8 \\ \Delta &\leq 144 \end{aligned}$$

$$\begin{aligned} 4 + \Delta/9 &\geq 0 \\ \Delta/9 &\geq -4 \\ \Delta &\geq -36 \end{aligned}$$

$$\begin{aligned} 48 - 2\Delta &\geq 0 \\ -2\Delta &\geq -48 \\ \Delta &\leq 24 \end{aligned}$$

Since

$$q_2 = 216 + \Delta$$

we have

$$\Delta = q_2 - 216$$

Substituting this value into the inequalities $\Delta \leq 144$, $\Delta \geq -36$, and $\Delta \leq 24$ gives a range of possible values for q_2 :

$$\begin{aligned} \Delta &\leq 144 \\ q_2 - 216 &\leq 144 \\ q_2 &\leq 360 \end{aligned}$$

$$\begin{aligned} \Delta &\geq -36 \\ q_2 - 216 &\geq -36 \\ q_2 &\geq 180 \end{aligned}$$

$$\begin{aligned} \Delta &\leq 24 \\ q_2 - 216 &\leq 24 \\ q_2 &\leq 240 \end{aligned}$$

That is,

$$180 \leq q_2 \leq 240 \leq 360$$