HARMONIC FUNCTIONS

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1.1 INTRODUCTION

Harmonic function is a mathematical function of two variables having the property that its value at any point is equal to the average of its values along any circle around that point, provided the function is defined within the circle. An infinite number of points are involved in this average, so that it must be found by means of an integral, which represents an infinite sum. In physical situations, harmonic functions describe those conditions of equilibrium such as the temperature or electrical charge distribution over a region in which the value at each point remains constant.

Harmonic functions can also be defined as functions that satisfy Laplace’s equation, a condition that can be shown to be equivalent to the first definition. The surface defined by a harmonic function has zero convexity, and these functions thus have the important property that they have no maximum or minimum values inside the region in which they are defined.

1.2 HARMONIC FUNCTIONS

Areal-valued functions H of two real variables x and y is said to be harmonic in a given domain of the xy plane if, through that the domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equations,

$$H_{xx}(x,y)+H_{yy}(x,y)=0$$

knows as Laplace's equation.

Theorem 1.2.1 If a function $f(z)=u(x,y)+iv(x,y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.

Proof:

Assuming that f is analytic in D, we start with the observation that the first order partial derivatives of its component functions must satisfy the Cauchy-Riemann equations throughout D.

$u_x = v_y$ and $u_y = -v_x$.

Differentiating both sides of these equations with respect to x we have:

$u_{xx} = v_{xy}$, $u_{xy} = -v_{xx}$.

The continuity of the partial derivatives of u and v ensures that $u_{yx} = u_{xy}$ and $v_{yx} = v_{xy}$.

Therefore $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

Example 1.2.2 Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

Proof:

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) + (-e^{-x})(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y)$$

$$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \quad \text{---------------------------- (1)}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x}\cos y - ye^{-x}\sin y - e^{-x}\cos y)$$

$$= -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y \quad \text{------------------------------ (2).}$$

Adding (1) and (2) yield

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence u is harmonic.

1.3 HARMONIC CONJUGATE

A function $u(x, y)$ defined on some open domain $\Omega \subset R^2$ is said to have as a conjugate function $v(x, y)$ if and only if they are respectively real and imaginary part of a holomorophic function $f(z)$ of the complex variable $z = x + iy \in \Omega$ That is, v is conjugated to u if $f(z)$ = $u(x, y)$ + $iv(x, y)$ is holomorphic on Ω. As a first consequence of the definition, they are both harmonic real-valued functions on Ω. Moreover, the conjugate of u, if it exists, is unique up to an additive constant. Also, u is conjugate to v if and only if v is conjugate to $-u$. Equivalently, v is conjugate to u in Ω if and only if u and v satisfy the Cauchy–Riemann equations in Ω. As an immediate consequence of the latter equivalent definition, if u is any harmonic function on $\Omega \subset \mathbb{R}^2$, the function u_y is conjugate to $-u_x$, for then the Cauchy–Riemann equations are just $\Delta u = 0$ and the symmetry of the mixed second order derivatives, $u_{xy} = u_{yx}$. Therefore an harmonic function u admits a conjugated harmonic function if and only if the holomorphic function $g(z)$: = $u_x(x,y) - iu_y(x,y)$ has a primitive $f(z)$ in Ω, in which case a conjugate of u is, of course, $-imf\ (x + iy)$. So any harmonic function always admits a conjugate function whenever its domain is simply connected, and in any case it admits a conjugate locally at any point of its domain.

There is an operator taking a harmonic function u on a simply connected region in R^2 to its harmonic conjugate v (putting e.g. $v(x_0)$ = 0 on a given x_0 in order to fix the indeterminacy of the conjugate up to constants). Conjugate harmonic functions (and the transform between them) are also one of the simplest examples of a Bäcklund transform (two PDEs and a transform relating their solutions), in this case linear; more complex transforms are of interest in solution and integral systems.

Geometrically u and v are related as having orthogonal trajectories, away from the zeroes of the underlying holomorphic function; the contours on which u and v are constant cross at right angles. In this regard, $u + iv$ would be the complex potential, where u is the potential function and v is the stream function.

Example 1.3.1 Consider the function $u(x,y)=e^x \sin y$

Solution:

Since $\frac{\partial u}{\partial x}=e^x \sin y$, $\frac{\partial^2 u}{\partial x^2}=e^x \sin y$

and $\frac{\partial u}{\partial y}=e^x \cos y$, $\frac{\partial^2 u}{\partial y^2}=-e^x \sin y$

It satisfies

$\Delta u=0.$

And thus is harmonic. Now suppose we have a $v(x,y)$ such that the Cauchy–Riemann equations are satisfied:

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=e^x \sin y$, and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=e^x \cos y$

Simplifying, $\frac{\partial v}{\partial y}=e^x \sin y$, and $\frac{\partial v}{\partial x}=-e^x \cos y$

which when solved gives $v=-e^x \cos y$.

Observe that if the functions related to u and v was interchanged, the functions would not be harmonic conjugates, since the minus sign in the Cauchy–Riemann equations makes the relationship asymmetric.

The conformal mapping property of analytic functions (at points where the derivative is not zero) gives rise to a geometric property of harmonic conjugates. Clearly the harmonic conjugate of x is y, and the lines of constant x and constant y are orthogonal. Co formality says that equally contours of constant $u(x,y)$ and $v(x,y)$ will also be orthogonal where they cross (away from the zeroes of $f'(z)$). That means that v is a specific solution of the orthogonal trajectory problem for the family of contours given by u (not the only solution, naturally, since we can take also functions of v): the question, going back to the mathematics of the seventeenth century, of finding the curves that cross a given family of non-intersecting curves at right angles.

There is an additional occurrence of the term harmonic conjugate in mathematics, and more specifically in geometry. Two points A and B are said to be harmonic conjugates of each other with respect to another pair of points C, D if $(ABCD) = -1$, where $(ABCD)$ is the cross-ratio of points A, B, C, D.

1.4 The Maxim Principle And The Mean Value Property

The Mean Value Property: suppose that if $u: U \to R$ is harmonic functions on an open set $U \subseteq C$ and that is a point $p_0 = \max_{z \in U} u(z)$,then u is a constant

$\overline{D}(0,1) \subseteq U$ for some $r > 0$. Then $u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p + re^{i\theta})d\theta$.

To understand why this result is true, let us simplify matters by assuming (with a simple translation of coordinates) that $p = 0$. Notice that if $k > 0$ and $u(z) = z^k$ then

$$u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p + re^{i\theta})d\theta \;\; p_{-}(\theta) = r^k \frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}d\theta = 0 = u(0).$$

The same hold when $k = 0$ by a similar calculation. So that is the mean value property for power of Z.

The Maximum Principle For Harmonic Functions: If $u: U \to R$ is a real –valued, harmonic function on a connected open set U and if there is a point $p_0 \in U$, with the property that $a \in D(0,1) = \min_{Q \in U} u(Q)$, then u is a constant on U.

The Minimum Principle Of Harmonic Functions: If $u: U \to R$ is a real-valued, harmonic function on a connected open set $U \subseteq C$ and if there is point $p_0 \in U$ such that $U(p_0) = \min_{Q \in U} u(Q)$, then u is consistent on U.

1.5 The Poisson Integral Formula

The next result shows how to calculate a harmonic function on the disc from its "boundary values" that is its values on the circle that bounds the disc.

Let u be a harmonic functions on a neighborhood of $\overline{D}(0,1)$ then, for any point

$a \in D(0,1)$ .Then $u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\psi}).\frac{1-|a^2|}{|a-e^{i\psi}|^2}\, d\psi.$

1.6 HARMONIC FUNCTION ON DISK

Before studying harmonic functions in the large it is necessary to study then locally. That is we must study these functions on disk. The plane is to study harmonic functions on the open unit disk $\{z : |z| < 1\}$.

What means unit disk? We learning the meaning unit disk, the open unit disk around P (where P is a given point in the plane) is the set of a point whose distance from P is less then 1.

$D_1(p) = \{Q : |p - Q| \leq 1\}$.

The closed unit disk around P is the set of points whose distant from P is less then or equal to one .

$\overline{D}(p) = \{Q : |p - Q| \leq 1\}$.

Poisson Kernel: Is important in a complex analysis because its integral against a function defined on a unit circle. The Poisson kernel commonly finds applications in control theory and two – dimensional problems in electrostatics.
The definition of Poisson kernels are often extended to n - dimension .

Definition 1.6.1 let D be the unit disc In the complex plane $\mathbb{C}$, the Poisson kernel for the unit disc D is given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = \operatorname{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right), \quad 0 \leq r < 1.$$

This can be thought of in two ways: either as a function of r and θ, or as a family of functions of θ indexed by r.

If $D = \{z : |z| < 1\}$ is the open unit disc in $\mathbb{C}$, T is the boundary of the disc, and f a function on T that lies in $L^1(T)$, then the function u given by

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\, \mathrm{d}t, \quad 0 \leq r < 1$$

is harmonic in D and has a radial limit that agrees with f almost everywhere on the boundary T of the disc.

Theorem 1.6.2 The function $p_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$

for $o \le r < 1$ and $-\infty < \theta < \infty$ is a poisson kernal.

Proof:

Let $z = re^{i\theta}$, $o \le r < 1$, then

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = (1+z)(1+z+z^2+...)$$

$$=1+2\sum_{n=1}^{\infty} z^n$$

Hence $\text{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = 1+2\sum_{n=1}^{\infty} r^n \cos n\theta$

$$=1+\sum_{n=1}^{\infty} r^n (e^{in\theta}+e^{-in\theta})$$
$$= p_{\_}(\theta)$$

$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1+re^{i\theta}-re^{-i\theta}-r^2}{\left|1-re^{i\theta}\right|^2}$ so that also

$$p_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2} = \text{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right).$$

Theorem 1.6.3 Let $D = \{z : |z| < 1\}$ and suppose that ∂D is continuous function. Then there is a continuous function $u : D \to R$ such that: (a) $u(z) = f(z)$ for z in ∂D, (b) u is harmonic in D more over u is unique and is defined by the formula:

$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p_r(\theta - t) f(e^{i\theta})dt$ for $0 \le r < 1$ and $0 \le \theta \le 2\pi$.

Proof:

Define $u : \overline{D} \to R$ by letting $u(re^{i\theta})$ in the above. If $0 \le r < 1$ and letting

$u(re^{i\theta})$ =$f(re^{i\theta})$ (i)u is harmonic in d. if $0 \le r < 1$ then

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) f(e^{i\theta})dt$$

$$= R\left\{\frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(e^{i\theta})\,\mathrm{Re}\left[\frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}}\right]dt\right\}$$

$$= R\left\{\frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(e^{i\theta})\left[\frac{e^{i\theta}+re^{i\theta}}{e^{i\theta}-re^{i\theta}}\right]dt\right\}$$

$$= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(e^{i\theta})\left[\frac{e^{it}+z}{e^{it}-z}\right]dt.$$

Hence u is continuous on $\overline{D}$.

Harnacks inequality: In mathematics, Harnacks inequality is an in equality relating the values of positive harmonic function at two points introduced by A. Harnack (1887) generalized Harnacks inequaliy to solution of elliptic or parabolic partial differential Equation.

Harnacks inequality is used to prove Harnacks theorem about the convergence of sequences of harmonic functions.

Let $D = D(z_0, R)$ be an open disk in the plane and let F be a harmonic functions on D, such that $f(z)$ is non-negative for all $z \in D$. Then the following inequality holds for all $z \in D$,

$$0 \le f(z) \le \left(\frac{R}{R-|z-z_0|}\right)^2 f(z_0),$$

for general domain Ω in R^n the inequality can be stated as follows: if ω is abounded domain with $\overline{\omega} \subset \Omega$, then there is a constant such that: $\sup_{x\to\omega} u(x) \le C \inf_{x\in\omega} u(x)$. For every twice differentiable, harmonic and non-negative functions $u(x)$. The constant C is independent of u.

In complex analysis, Harnacks principle or Harnacks theorem about the convergent of sequences of harmonic functions that follow from Harnacks inequality.

If the function $u_1(z), u_2(z), \ldots$ are harmonic in an $u_1(z) \le u_2(z), \ldots$ every point of G, then the limit, $\lim_{n\to\infty} u_n(z)$ either is an infinite in every point of the domain G or it is infinite in every point of the domain, in both cases uniformly in each compact subset of G.

Weierstrass approximation theorem: states that every continues functions defined on an interval $[a,b]$ can be uniformly approximated as closely as desired by a polynomial functions.

The statement of the approximation theorem as originally discovered by the Weierstrass is as follows: suppose F is a continuous complex–valued function defined on the real interval $[a,b]$, for every $\varepsilon > 0$ there exists a polynomial functions p over $\mathbb{C}$ such that for all x in $[a,b]$, we have $|f(x) - p(x)| < \varepsilon$ or equivalent, the supermom norm $\|f\ (x) - p\| < \varepsilon$. If f is real valued, the polynomial function can be taken over $\mathbb{R}$.

Definition 1.6.4 Any continuous functions on abounded interval can be uniformly approximated by polynomial functions.

Theorem 1.6.5 Let $U \subseteq C$ be any open set. let $a_1, a_2, \ldots$ be a finite or infinite sequence in U (possibly with repetitions) that has no accumulation point in U. Then there exists a holomorphic function f on U whose zero set is precisely (a_j).

The function f is constructed by taking an infinite product. The proof converts the problem to a situation on the entire plan, and then uses the Weierstrass product. We next want to formulate a result about maximal domains of existence (or domains of definition) of holomorphic functions.

But first we need a geometric fact about open subset of the plane.

1.7 SUBHARMONIC FUNCTIONS AND SUPERHARMONIC FUNCTIONS

What is a sub harmonic function?. Subharmonic functions Areal- valued function u whose domains a two-dimensional domain D is subharmonic in a D provided u satisfies the following conditions in D :

1- $-\infty \leq u(x,y) < \infty$ where $u(x,y) \neq -\infty$.

2- u is upper semicontinuouse in D.

3- For any sub domain D' included together with its boundary B', in D and for any function h harmonic in D' continuous in $D' + B'$, and satisfying $h(x,y) \geq u(x,y)$ on B' we have $h(x,y) \geq u(x,y)$ in D'.

Subharmonic functions can be defined by the same description, only replacing“ no larger ” with“ no smaller”. Alternatively, a subharmonic function is just negative of a sub harmonic function, and for these reason and property of subharmonic functions can be easily transferred to super harmonic function.

Formally, the definition can be stated as follows. Let G be a subset of the R^n and let $\varphi : G \to R \cup (-\infty)$ be an upper semi-continuous function. Then, φ is called sub harmonic if for any closed ball $\overline{B(x,R)}$ of centre x and radius r contained in G and every real-valued continuous function h on $\overline{B(x,R)}$ that is harmonic in $B(x,r)$ and satisfies $\varphi(x) \leq h(x)$ for all x on the boundary $\partial B(x,r)$ of $B(x,r)$ we have $\varphi(x) \leq h(x)$ for all $x \in B(x,r)$. Note that by the above, the function which is identically $-\infty$ is subharmonic, but some authors exclude this function by definition.

Definition 1.7.1 Let $G \subset C$ be a region and let $\varphi : G \to R$ be a continuous function φ is said to be subharmonic if whenever $D(z,r) \subset C$ (where $D(z,r)$ is closed disc around z of radius r) we have

$$\varphi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + re^{i\theta}) d\theta$$

and φ is said to be superharmonic if whenever $D(z,r) \subset G$ we have

$$\varphi(z) \geq \frac{1}{2\pi}\int_0^{2\pi} \varphi(z + re^{i\theta})d\theta.$$

Intuitively what this means is that a subharmonic function is at any point no greater than the average of the values in a circle around that point. This implies that a non-constant subharmonic function does not achieve its maximum in a region G (it would achieve it at the boundary if it is continuous). Similarly for a superharmonic function, but then a non-constant superharmonic function does not achieve its minimum in G. It is also easy to see that φ is subharmonic if and only if $-\varphi$ is superharmonic.

Note that when equality always holds in the above equation then φ would in fact be a harmonic function. That is, when φ is both subharmonic and superharmonic, then φ is harmonic.

A function $h(z) = h(x, y)$ is said to be subharmonic on adomain G if it has the following properties:

1- $h(x, y)$ is defined continuous at every point of G except possibly at a finite number of points or at points of a sequence $\{(a_n, b_n)\}$ which has no limit points in G, while every exceptional point (a_n, b_n).

Satisfies the relation $\lim_{(x,y)\to(a_n,b_n)} h(x, y) = -\infty$ which is used to define $h(a_n, b_n) = -\infty$

2- The integral $\frac{1}{2\pi}\int_0^{2\pi} h(z_0 + \rho e^{i\varphi})d\varphi$ exists for every point $z_0 \in G$ and all ρ less then $D(z_0)$ the distance between z_0 and the boundary of G.

3- The inequality $h(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + \rho e^{i\varphi})d\varphi$ holds for every point $z_0 \in G$ and all sufficiently small $\rho > 0$.

Theorem 1.7.2 Let Ω be open in C and let u be use on Ω, suppose that u is not identically $-\infty$ on any connected component of Ω.

1- If u is subharmonic on Ω , and if $a \in \Omega$ and $R > 0$ are such that u then $u(z) \le P_{a,r}(u)(z)$ for all $z \in D$ where $P_{a,r}(u)$ is the Poisson integral of u $u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a + \mathrm{Re}^{it})dt$

2- if, for every $a \in \Omega$ there is $R_a > 0$ such that $u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a + \mathrm{Re}^{it})dt$ for $0 < R < R_a$. Then u is subharmonic on Ω.

Proof:

part1. Let $(\varnothing_n)_{n\ge1}$ be a sequence of continuous functions on $\overline{D(a,R)}$ such that $\phi_n \downarrow u$.
Let $h_n = p_{a,R}(\phi_n)$ then h_n is continuous on $\overline{D(a,R)}$ and harmonic $D(a,R)$ and $h_n = \phi_n \ge u$ on $\partial D(a,R)$. Hence, if u is harmonic, we have $u(z) \le h_n(z) = \frac{1}{2\pi}\int_0^{2\pi} P_{a,R}(z,t)\phi_n(a + \mathrm{Re}^{it})dt$. Now $P_{a,r}(z,t) > 0$ and $\phi_n \downarrow u$. Hence by the monotone convergence.

Part2. Let $U \subseteq \Omega$ let h be continuous on $\overline{U}$ and harmonic on U. Suppose that $u \le h$ on ∂U. Given $\varepsilon > 0$, every boundary point $a \in \partial U$ has neighborhood D_a such that $u - h < \varepsilon$ on D_a. Let $z \in U$, and let V be an open set such that $z \in V$, $V \subseteq U$ and $\partial V \subset \bigcup_{a \in \partial U}^{n} D_a$. Hence $u(z) - h(z) \le \sup_{w \in \partial V}(u(w) - h(w)) \le \varepsilon.$

Since $z \in U$ and $\varepsilon > 0$ are arbitrary, we have $u - h \le 0$ on U and u is subharmonic.

The Dirichlet Problem: Definition: A region G is called a Dirichlet Region if the Dirichlet problem can be solved by G, that is G is a Dirichlet Region if for each continuous function $f : \partial_\infty G \to R$, there is continuous function $u : \overline{G} \to R$ such that u is harmonic in G and $u(z) = f(z)$ for all z in $\partial_\infty G$.

Theorem 1.7.3 Let f be a continuous function on define

$$u(z) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\psi}).\frac{1 - |z|^2}{|z - \mathrm{e}^{i\psi}|^2} d\psi, & \text{if } z \in D(0,1) \\ f(z), & \text{if } z \in \partial D(0,1) \end{cases}$$

Then u is continuous on $\overline{D(0,1)}$ and harmonic on $D(0,1)$. closely related to this result is the reproducing proper of the Poisson kernel: let u be continuous on $\overline{D(0,1)}$ and harmonic on $D(0,1)$. Then, for $z \in D(0,1)$

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi})\frac{1-|z|^2}{|z-e^{i\psi}|}d\psi.$$

Some of the basic questions in regard harmonic functions are related to the so-called Dirichlet problem, which we frame in the following ways: Given a region $D \subset Z$, let p be a continuous real-valued function whose domain is ∂D. Determine a function (or all function) that is continuous in $\overline{D}$ Harmonic in D, and whose restriction to ∂D is p.

The Dirichlet Problem On A General Disc: A change variable shows that the result of remain true on a general disc. Let f be a continuous on $\partial D(P,R)$.

Define $$u(z) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\psi}).\frac{R-|z-P|^2}{|(z-P)-R\,e^{i\psi}|^2}d\psi, & \text{if } z \in D(P,R) \\ f(z), & \text{if } z \in \partial D(P,R) \end{cases}$$

Then u is a continuous on $\overline{D(P,R)}$ and harmonic on $D(P,R)$. If instead u is harmonic on a neighborhood of $\overline{D(P,R)}$ then, for $z \in D(P,R)$.

Application Of Conformal Mapping To The Dirichlet Problem: Let $\Omega \subseteq C$ be a domain whose boundary consists of finitely many smooth curves. The dirichlet problem which mathematical problem of interest in its own right, is the boundary value problem $\Delta u = 0$ on Ω $u = f$ on $\partial\Omega$.

The way to think about this problem is a follows: a data a function. on the boundary of the domain is given to solve the corresponding Dirichlet problem, one seeks a continuous function u on the closure of U (that is, the union of U andits boundary) such that u is harmonic on, and agrees with, on the boundary. We shall now describe three distinct physical situations that are mathematically modeled by the Dirichlet problem.

Heat Diffusion: I imagine that Ω Is a thin plat of heat –conducting metal. The shape of Ω is arbitrary (not necessarily a rectangle) see figure I A function $u(x,y)$ describes the temperature at each point (x,y) in Ω. It is standard situation in engineering or physics to consider idealized heat sources or sinks that maintain specified (fixed) clause of u on certain parts of the boundary, other parts of the boundary are to be thermally insulated. One wants to find the steady states heat distribution on Ω (that is, as to $t \to +\infty$) that is determined by the given boundary condition.

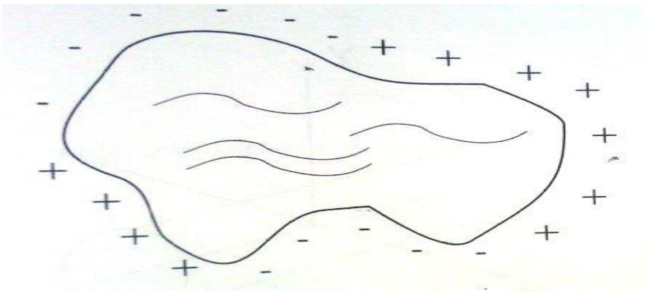

Figure I

Electrostatic Potential**:** Now we describe a situation in electrostatics that is modeled by the boundary value.

Imagine a long, bellow the cylinder made of a thin sheet of some conducting material, such as copper. split the cylinder lengthwise into two equal pieces figure II separate the two pieces with strips of insulating material. Now ground the upper of the semi-cylinder pieces to potential zero, and keep the lower piece at some non zero fixed potential. for simplicity in the present discussion, let us say that this last fixed potential is 1.

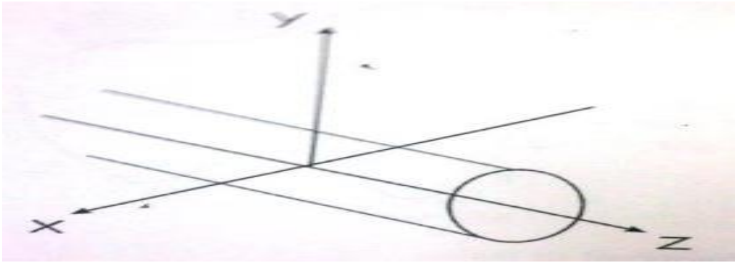

Figure II

1.8 CONCLUSION

The title of this project is harmonic functions it has very interesting and help full for solve many problems are important in the areas of applied mathematics, engineering, and mathematical physics. Harmonic functions are used to solve problems involving steady state temperatures, two-dimensional electrostatics, and ideal fluid flow.

We get some difficult from this title, so this title to facilitate for the students of the degree and graduate of the university to concept from this title of this project. Because this research consist basic, definition, examples, demonstrated a among this title of this project which Can make the reader to get some concepts and advantages from this title of this project.

Finally we will encourage students to read this title from various books, because the concept of harmonic function, subharmonic function and subharmonic functions of the fundamental ideas.

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