Lecture 6

Linear programming :
Artificial variable technique :
Big - M method

6.1 Computational Procedure of Big – M Method (Charne's Penalty Method)

Step 1 – Express the problem in the standard form.

Step 2 – Add non-negative artificial variable to the left side of each of the equations corresponding to the constraints of the type '$\geq$' or '='.

When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists).

On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by –M for maximization problems (+M for minimizing problem), where M > 0.

Step 3 – In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

6.2 Worked Examples

Example 1

Max $Z = -2x_1 - x_2$

Subject to

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Max $Z = -2x_1 - x_2 + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1 , x_2 , s_1, s_2, a_1, a_2 \geq 0$

	$C_j \rightarrow$		-2	-1	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-M	3	[3]	1	0	0	1	0	3 /3 = 1→
a_2	-M	6	4	3	-1	0	0	1	6 / 4 =1.5
s_2	0	4	1	2	0	1	0	0	4 / 1 = 4
	Z = -9M		↑ 2 – 7M	1 – 4M	M	0	0	0	←Δ_j
x_1	-2	1	1	1/3	0	0	X	0	1/1/3 =3
a_2	-M	2	0	[5/3]	-1	0	X	1	6/5/3 =1.2→
s_2	0	3	0	5/3	0	1	X	0	4/5/3=1.8
	Z = -2 – 2M		0	↑ $\frac{(-5M+1)}{3}$	0	0	X	0	←Δ_j
x_1	-2	3/5	1	0	1/5	0	X	X	
x_2	-1	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	Z = -12/5		0	0	1/5	0	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = -12/5, $x_1 = 3/5$, $x_2 = 6/5$

Example 2

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Max $Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - M a_1$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 3$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

		$C_j \rightarrow$	3	-1	0	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-M	2	[2]	1	-1	0	0	1	$2 / 2 = 1 \rightarrow$
s_2	0	3	1	3	0	1	0	0	$3 / 1 = 3$
s_3	0	4	0	1	0	0	1	0	-
	Z = -2M		$\uparrow$ -2M-3	-M+1	M	0	0	0	$\leftarrow \Delta_j$
x_1	3	1	1	1/2	-1/2	0	0	X	-
s_2	0	2	0	5/2	[1/2]	1	0	X	$2/1/2 = 4 \rightarrow$
s_3	0	4	0	1	0	0	1	X	-
	Z = 3		0	5/2	$\uparrow$ -3/2	0	0	X	$\leftarrow \Delta_j$
x_1	3	3	1	3	0	1/2	0	X	
s_1	0	4	0	5	1	2	0	X	
s_3	0	4	0	1	0	0	1	X	
	Z = 9		0	10	0	3/2	0	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 9, $x_1 = 3$, $x_2 = 0$

Example 3

Max $Z = 3x_1 + 2x_2 + x_3$

Subject to

$2x_1 + x_2 + x_3 = 12$

$3x_1 + 4x_2 = 11$

and x_1 is unrestricted

$x_2 \geq 0,\ x_3 \geq 0$

Solution

SLPP

Max $Z = 3(x_1' - x_1'') + 2x_2 + x_3 - M\,a_1 - M\,a_2$

Subject to

$2(x_1' - x_1'') + x_2 + x_3 + a_1 = 12$

$3(x_1' - x_1'') + 4x_2 + a_2 = 11$

$x_1',\ x_1'',\ x_2,\ x_3,\ a_1,\ a_2 \geq 0$

Max $Z = 3x_1' - 3x_1'' + 2x_2 + x_3 - M\,a_1 - M\,a_2$

Subject to

$2x_1' - 2x_1'' + x_2 + x_3 + a_1 = 12$

$3x_1' - 3x_1'' + 4x_2 + a_2 = 11$

$x_1',\ x_1'',\ x_2,\ x_3,\ a_1,\ a_2 \geq 0$

		$C_j \rightarrow$	3	-3	2	1	-M	-M	
Basic Variables	C_B	X_B	X_1'	X_1''	X_2	X_3	A_1	A_2	Min ratio X_B / X_k
a_1	-M	12	2	-2	1	1	1	0	12 /2 = 6
a_2	-M	11	[3]	-3	4	0	0	1	11/3 =3.6→
	Z = -23M		↑ -5M-3	5M+3	-5M-2	-M-1	0	0	←Δ_j
a_1	-M	14/3	0	0	-5/3	[1]	1	X	14/3/1 = 14/3→
x_1'	3	11/3	1	-1	4/3	0	0	X	-
	$Z = \frac{-14M+11}{3}$		0	-6	5/3M+1	↑ -M-1	0	X	←Δ_j
x_3	1	14/3	0	0	-5/3	1	X	X	
x_1'	3	11/3	1	-1	4/3	0	X	X	
	Z = 47/3		0	0	1/3	0	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

$x_1' = 11/3,\ \ x_1'' = 0$

$x_1 = x_1' - x_1'' = 11/3 – 0 = 11/3$

Therefore the solution is Max Z = 47/3, $x_1 = 11/3$, $x_2 = 0$, $x_3 = 14/3$