Lecture 11

Linear programming :
The Revised Simplex Method

11.1 The Revised Simplex Method

While solving linear programming problem on a digital computer by regular simplex method, it requires storing the entire simplex table in the memory of the computer table, which may not be feasible for very large problem. But it is necessary to calculate each table during each iteration. The revised simplex method which is a modification of the original method is more economical on the computer, as it computes and stores only the relevant information needed currently for testing and / or improving the current solution. i.e. it needs only

- The net evaluation row Δ_j to determine the non-basic variable that enters the basis.
- The pivot column
- The current basis variables and their values (X_B column) to determine the minimum positive ratio and then identify the basis variable to leave the basis.

The above information is directly obtained from the original equations by making use of the inverse of the current basis matrix at any iteration.

There are two standard forms for revised simplex method

- **Standard form-I –** In this form, it is assumed that an identity matrix is obtained after introducing slack variables only.

- **Standard form-II –** If artificial variables are needed for an identity matrix, then two-phase method of ordinary simplex method is used in a slightly different way to handle artificial variables.

11.2 Steps for solving Revised Simplex Method in Standard Form-I

Solve by Revised simplex method
Max $Z = 2x_1 + x_2$
Subject to

$3x_1 + 4x_2 \le 6$
$6x_1 + x_2 \le 3$

and $x_1, x_2 \ge 0$

SLPP
Max $Z = 2x_1 + x_2 + 0s_1 + 0s_2$
Subject to

$3x_1 + 4x_2 + s_1 = 6$
$6x_1 + x_2 + s_2 = 3$

and $x_1, x_2, s_1, s_2 \ge 0$

Step 1 – Express the given problem in standard form – I

- Ensure all $b_i \geq 0$
- The objective function should be of maximization
- Use of non-negative slack variables to convert inequalities to equations

The objective function is also treated as first constraint equation

$$
\begin{aligned}
&Z - 2x_1 - x_2 + 0s_1 + 0s_2 = 0 \\
&3x_1 + 4x_2 + s_1 + 0s_2 = 6 \qquad\qquad \text{-- (1)} \\
&6x_1 + x_2 + 0s_1 + s_2 = 3
\end{aligned}
$$

and $x_1, x_2, s_1, s_2 \geq 0$

Step 2 – Construct the starting table in the revised simplex form

Express (1) in the matrix form with suitable notation

$$
\begin{array}{ccccc}
\beta_0^{(1)} & & & \beta_1^{(1)} & \beta_2^{(1)} \\
e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)}
\end{array}
\qquad\qquad\qquad X_B
$$

$$
\begin{bmatrix} 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 \end{bmatrix}
\begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}
$$

Column vector corresponding to Z is usually denoted by e_1 matrix B_1, which is usually denoted as $B_1 = [\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)} \ldots \beta_n^{(1)}]$

Hence the column $\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}$ constitutes the basis matrix B_1 (whose inverse B_1^{-1} is also B_1)

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0		
s_1	0	1	0	6		
s_2	0	0	1	3		

$a_1^{(1)}$	$a_2^{(1)}$
-2	-1
3	4
6	1

Step 3 – Computation of Δ_j for $a_1^{(1)}$ and $a_2^{(1)}$

Δ_1 = first row of $B_1^{-1} * a_1^{(1)} = 1 * -2 + 0 * 3 + 0 * 6 = -2$

Δ_2 = first row of $B_1^{-1} * a_2^{(1)} = 1 * -1 + 0 * 4 + 0 * 1 = -1$

Step 4 – Apply the test of optimality

Both Δ_1 and Δ_2 are negative. So find the most negative value and determine the incoming vector.

Therefore most negative value is $\Delta_1 = -2$. This indicates $a_1^{(1)}$ (x_1) is incoming vector.

Step 5 – Compute the column vector X_k

$X_k = B_1^{-1} * a_1^{(1)}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$$

Step 6 – Determine the outgoing vector. We are not supposed to calculate for Z row.

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0	-2	-
s_1	0	1	0	6	3	2
s_2	0	0	1	3	$\boxed{6}$ ↑ incoming	1/2→outgoing

Step 7 – Determination of improved solution

Column e_1 will never change, x_1 is incoming so place it outside the rectangular boundary

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_1
R_1	0	0	0	-2
R_2	1	0	6	3
R_3	0	1	3	$\boxed{6}$

Make the pivot element as 1 and the respective column elements to zero.

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_1
R_1	0	1/3	1	0
R_2	1	-1/2	9/2	0
R_3	0	1/6	1/2	1

Construct the table to start with second iteration

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	1/3	1		
s_1	0	1	-1/2	9/2		
x_1	0	0	1/6	1/2		

$a_4^{(1)}$	$a_2^{(1)}$
0	-1
0	4
1	1

$\Delta_4 = 1 * 0 + 0 * 0 + 1/3 *1 = 1/3$

$\Delta_2 = 1 * -1 + 0 * 4 + 1/3 *1 = -2/3$

Δ_2 is most negative. Therefore $a_2^{(1)}$ is incoming vector.

Compute the column vector

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{bmatrix} * \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 7/2 \\ 1/6 \end{bmatrix}$$

Determine the outgoing vector

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	1/3	1	-2/3	-
s_1	0	1	-1/2	9/2	[7/2]	9/7→outgoing
x_1	0	0	1/6	1/2	1/6	3
					↑ incoming	

Determination of improved solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_2
R_1	0	1/3	1	-2/3
R_2	1	-1/2	9/2	[7/2]
R_3	0	1/6	1/2	1/6

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_2
R_1	4/21	5/21	13/7	0
R_2	2/7	-1/7	9/7	1
R_3	-1/21	8/42	2/7	0

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	4/21	5/21	13/7		
x_2	0	2/7	-1/7	9/7		
x_1	0	-1/21	8/42	2/7		

$a_4^{(1)}$	$a_3^{(1)}$
0	0
0	1
1	0

$\Delta_4 = 1 * 0 + 4/21 * 0 + 5/21 *1 = 5/21$
$\Delta_3 = 1 * 0 + 4/21 * 1 + 5/21 *0 = 4/21$

Δ_4 and Δ_3 are positive. Therefore optimal solution is Max Z = 13/7, x_1= 2/7, x_2 = 9/7

11.3 Worked Examples

Example 1

Max $Z = x_1 + 2x_2$
Subject to

$$x_1 + x_2 \le 3$$
$$x_1 + 2x_2 \le 5$$
$$3x_1 + x_2 \le 6$$

and $x_1, x_2 \ge 0$

Solution

SLPP

Max $Z = x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3$
Subject to

$$x_1 + x_2 + s_1 = 3$$
$$x_1 + 2x_2 + s_2 = 5$$
$$3x_1 + x_2 + s_3 = 6$$

and $x_1, x_2, s_1, s_2, s_3 \ge 0$

Standard Form-I

$$Z - x_1 - 2x_2 - 0s_1 - 0s_2 - 0s_3 = 0$$
$$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 3$$
$$x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 5$$
$$3x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 6$$

and $x_1, x_2, s_1, s_2, s_3 \ge 0$

Matrix form

$$\begin{array}{cccccc} \beta_0^{(1)} & & & \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 6 \end{bmatrix}$$

Revised simplex table

Basic variables	B_1^{-1} e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X_B	X_k	X_B / X_k
Z	1	0	0	0	0		
s_1	0	1	0	0	3		
s_2	0	0	1	0	5		
s_3	0	0	0	1	6		

Additional table

$a_1^{(1)}$	$a_2^{(1)}$
-1	-2
1	1
1	2
3	1

Computation of Δ_j for $a_1^{(1)}$ and $a_2^{(1)}$

Δ_1 = first row of $B_1^{-1} * a_1^{(1)} = 1 * -1 + 0 * 1 + 0 *1 + 0 *3= -1$
Δ_2 = first row of $B_1^{-1} * a_2^{(1)} = 1 * -2 + 0 * 1 + 0 *2+ 0 *1 = -2$

$\Delta_2 = -2$ is most negative. So $a_2^{(1)}$ (x_2) is incoming vector.

Compute the column vector X_k

$$X_k = B_1^{-1} * a_2^{(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Basic variables	B_1^{-1}				X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$			
Z	1	0	0	0	0	-2	-
s_1	0	1	0	0	3	1	3
s_2	0	0	1	0	5	[2]	5/2→
s_3	0	0	0	1	6	1	6
						↑	

Improved Solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X_B	X_k
R_1	0	0	0	0	-2
R_2	1	0	0	3	1
R_3	0	1	0	5	[2]
R_4	0	0	1	6	1

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X_B	X_k
R_1	0	1	0	5	0
R_2	1	-1/2	0	1/2	0
R_3	0	1/2	0	5/2	1
R_4	0	-1/2	1	7/2	0

Revised simplex table for II iteration

Basic variables	B_1^{-1}				X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$			
Z	1	0	1	0	5		
s_1	0	1	-1/2	0	1/2		
x_2	0	0	1/2	0	5/2		
s_3	0	0	-1/2	1	7/2		

$a_1^{(1)}$	$a_4^{(1)}$
-1	0
1	0
1	1
3	0

$\Delta_1 = 1 * -1 + 0 * 1 + 1 * 1 + 0 * 3 = 0$
$\Delta_4 = 1 * 0 + 0 * 0 + 1 * 1 + 0 * 0 = 1$

Δ_1 and Δ_4 are positive. Therefore optimal solution is Max Z = 5, $x_1 = 0$, $x_2 = 5/2$