Eigen values and Eigen functions

Another set of questions with great physical significance can be addressed. What properties of a system are quantized, what are not, and why? If a property is quantized, what possible results will measurements of such a property yield? These questions can now be answered precisely mathematically. The allowed values of any property (or the result of any measurement of the property) are limited to the eigen values of the operator representing this property. If the corresponding eigen values are discrete, this property is quantized; otherwise, it is not. What, then, are the ‘‘eigen values’’ and ‘‘eigen functions’’ of an operator? (‘‘Eigen’’ came from the German word ‘‘Eigentum’’ that does not seem to have a precise English translation. It means something like ‘‘characteristic’’ or ‘‘distinct,’’ or more precisely the ‘‘idio’’ part of ‘‘idiosyncrasy’’ in Greek, but its precise interpretation is probably best inferred from how it is used in context.)

(Eigen function is give up the same function since operating by mathematical operator and the attached constant value is Eigen value).

As stated earlier, in general an operator operating on an arbitrary state function will change it to another state function. It can be shown that, associated with each operator representing a physically observable property, there is a unique set of characteristic state functions that will not change when operated upon by the operator. These state functions are called the ‘‘eigen functions’’ of this operator. Application of such an operator on each of its eigen functions leads to a characteristic number, which is a real number (no imaginary part), multiplying this eigen function. The characteristic number corresponding to each eigen function of the operator is called the ‘‘eigen value’’ corresponding to this eigen function. For example, the eigen value equation with discrete eigen values:

$$\hat{A}(\chi_i)\, f(\chi_i) = p f(\chi_i) \qquad \text{--------(11)}$$

Eq(11) gives the effect of an operator $\hat{A}(\chi_i)$ on its eigen function $f(\chi_i)$ corresponding to the eigen value p. For continuous eigen values: where χ_i is a continuous variable. For some operators, some of the eigen values are discrete while the others are continuous. Accordingly, the problem of determining the allowed values of any property of the system is now reduced to that of solving the eigen value

Since $\hat{A}(\chi_i)$ is mathematical operator , $f(\chi_i)$ is function, and p is Eigen value.

Examples:-

$$, \quad \frac{d2}{dx2}(sinax) = , \quad \frac{d}{dx}(\frac{d}{dx}sinax) = (\frac{d}{dx}acosax)$$

$$= -a2sinax \quad \text{therefore}$$

$sinax$ is Eigen function and $-a^2$ is Eigen value

$$\frac{d2}{dx2}(cosax) = , \quad \frac{d}{dx}(\frac{d}{dx}cosax) = (\frac{d}{dx}asinax)$$

$$= -a2sinax \quad \text{therefore}$$

$sinax$ is Eigen function and $-a^2$ is Eigen value

Example1 // explain if that's function $sin2x$ is Eigen function for the following mathematical operator $(cosx\frac{d3}{dx3})$,and $(tanx\frac{d3}{dx3})$,what is eigen values if it's found.

Solution//

$$cosx\frac{d3}{dx3}(sin2x) = cosx\frac{d2}{dx2}(\frac{d}{dx}sin2x)$$

$$= 2cosx\frac{d}{dx}(\frac{d}{dx}sinxcosx)$$

$$= 2cosx\frac{d}{dx}(-sinx.sinx + cosx.cosx)$$

$$= 2cosx\frac{d}{dx}(cos2x - sin2x)$$

$$= 2cosx(-2cosxsinx - 2cosx(2sinxcosx)$$

$$= -4cos2xsinx - 4cos2xsinx$$

$= -8cos2x.sinx$ the function $sin2x$ is not eigen function for this mathematical operator.

$$tanx\frac{d3}{dx3}(sin2x) = tanx\frac{d2}{dx2}(\frac{d}{dx}sin2x)$$

$$= 2tanx\frac{d}{dx}(\frac{d}{dx}sinxcosx)$$

$$= 2tanx\frac{d}{dx}(-sinx.sinx + cosx.cosx)$$

$$= 2tanx\frac{d}{dx}(cos2x - sin2x\ )$$

$$= 2tanx(-2cosxsinx - 2tanx(2sinxcosx\ )$$

$$= -4\left(\frac{sinx}{cosx}\right).cosx.sinx - 4\left(\frac{sinx}{cosx}\right).cosx.sinx$$

$= -8sin2x$ the function $sin2x$ is eigen function for this mathematical operator and the eigen value is -8.

Home work:-

1- $\frac{d2}{dx2}$ for e^{-ax}

2- $cosx.\frac{d3}{dx3}\quad for\ \ cosx$

3- ∇^2 for cosax . cosby .coscz

4- $-h2/4\pi2(\frac{d2}{dx2} + cotx\frac{d}{dx})\,for\ \ 3cos2x - 1.$

<u>Mathematical and physical Explanation of wave function</u>

The definition system by Schrödinger equation has double character phenomena (particle-wave). The wave function Ψ is the value of peak or the wave capacity and Schrödinger give it another name , it's a measurements of charge density around the space of nuclei.

The determine mathematical explanation represented by:-

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1. The function must be finite and limited in anywhere of the space , the wave approach to zero when the dimension approach to infinity.

2. The function must be has single value because the probability of electron foundation in the nuclei space must be equal to one.

3. The function must be continuously for all physical values of dimension to give up suitable salvations of wave function.

Postulates of Quantum mechanics

1- The probability of electron foundation in anywhere has single value along the axis of dimension.

2- Any physical quantity for a chemical system must be represented by mathematical operator (linear operator).

3- The wave function $\Psi(q,t)$ is the Schrödinger equation depend on the time ,since it determine the past and present state of system if the present state is known .$\hat{H}\,\Psi(q,t)=h/2\pi i\ .\partial/\partial t\ .\ \Psi(q,t)$

4- If a single measurements has been done on the system according to mathematical operator $\hat{H}$, the only mathematics value is the associated value to the function $\hat{H}\,\Psi=a\,\Psi$.

5- If $\Psi 1$ and $\Psi 2$ are eigen functions for the eigen values a1 and a2 respectively for the mathematic operator $\hat{H}$, they found another salvations are borne from linear combination of Ψ_1 and Ψ_2 , its Ψ as following

$\Psi = C1\Psi 1 + C2\ \Psi 2$ since C1 and C2 are constant quantity for system state to give finite probability to measurable a1 and a2 respectively.

<u>Hermitic character of mathematical operators</u>

An interesting question that can now be addressed is this. How does one know
whether it is possible to have complete simultaneous knowledge of two specific
properties of a system, say ‘‘A’’ and ‘‘B’’?

Physically, for two properties to be specified simultaneously, it must be possible to measure one of the two properties without influencing the outcome of the measurement of the other property, and vice versa (العكس بالعكس). In short, the order of measurements of the two properties should not matter, no matter what state the system is in. This means that application of the operator $\hat{A}\,\hat{S}$ on any arbitrary state C should be exactly the same as applying the operator $\hat{A}\,\hat{S}$ on the same state, or: $(\hat{A}\hat{S}\ \Psi) = (\hat{S}\hat{A}\ \Psi)$ **-------12**

It in turn means that the effect of the operator **($\hat{A}\,\hat{S}$ = $\hat{S}\,\hat{A}$)**on any(تحكمي)arbitrary state of the system must be equal to zero in this case:**(($\hat{A}\,\hat{S}$ - $\hat{S}\,\hat{A}$) $\Psi = 0$)**

The operator itself is, therefore, equivalent to a ‘‘null operator’’:**($\hat{A}\,\hat{S}$ - $\hat{S}\,\hat{A}$)** = 0

when applied to any arbitrary state of the system, if the two properties can be specified precisely simultaneously. The difference of two operators applied in different order is called the ‘‘commutated’’(المحول).

When the commutator of two operators is equal to zero, the two operators are said to ‘‘commute.’’ When two operators commute, as **$\hat{A}$ and $\hat{S}$** in Eq. it means that the two corresponding dynamic properties of the system can be measured in arbitrary order and specified precisely simultaneously, regardless (مهمل)of what state the system is in. There is now, therefore, a mathematically rigorous way to determine which two physical properties can be specified simultaneously and which ones may not be by simply calculating the commutator of the two corresponding operators.

Degenerates and Orthogonal functions

Expansion of wave functions:-

There are a common applications for Expansion of wave functions in the problems of quantum mechanics, since the general wave function (Ψ) is represent the linear combination for completely groups of orthonormal functions as in following representation:-

$$\Psi = C_1 \Psi_1 + C_2 \Psi_2 + C_3 \Psi_3 + C_4 \Psi_4 + \ldots\ldots\ldots\ldots + C_n \Psi_n \text{ ------}(1)$$

Since the C_1 ,C_2 ,C_3 ,C_4 ,C_n are constants, and eq-1 are equivalent with the equation of direction analysis by its components along the dimensions .to find the constants must be multiple eq-1 by Ψ^*_j

J=1 into n values.

$$\Psi^*_j \Psi = C_1 \Psi^*_j \Psi_1 + C_2 \Psi^*_j \Psi_2 + C_3 \Psi^*_j \Psi_3 + C_4 \Psi^*_j \Psi_4 ++ C_n \Psi^*_j \Psi_n \ (2)$$

By integral eq-2 according to the element of volume.

$$\int\Psi^*_j \Psi \partial\tau = C_1 \int\Psi^*_j \Psi_1 \partial\tau + C_2 \int\Psi^*_j \Psi_2 \partial\tau + C_3 \int\Psi^*_j \Psi_3 \partial\tau +$$

$$+C_4 \int\Psi^*_j \Psi_4 \partial\tau + \ldots\ldots\ldots\ldots + C_n \int\Psi^*_j \Psi_n \partial\tau \text{------ } (3)$$

At the right side of eq-3 „j will take the same values of wave functions ($\Psi^*_j = \Psi^*_1$) and so on for each terms and there for all integrals in eq-3 equal to 1 due normalization condition. If the values are differ ,all integrals equal to zero due orthogonal condition.

$$\int \Psi^*_j \Psi \partial\tau = C_j \text{ --------(4)}$$

eq-4 represent the value of expansion coefficient and can be using this equation to find the **average value of unobservable quantity** .

if we supposed that Ψ_1 ,Ψ_2 ,Ψ_3 ,Ψ_4 ,and Ψ_n are completely groups of orthonormal functions for the mathematical operator Â that's a suitable for observed value in this system. So that if eq-1 acted by Â this produced :-

$$\hat{A}\,\Psi = C_1\,\hat{A}\,\Psi_1 + C_2\,\hat{A}\,\Psi_2 + C_3\,\hat{A}\,\Psi_3 + C_4\,\hat{A}\,\Psi_4 + \ldots\ldots + C_n\,\hat{A}\,\Psi_n \text{ ------(5)}$$

According to fourth hypothesis $\hat{A}\,\Psi = a\,\Psi$ so that

$$\hat{A}\,\Psi = C_1\,a_1\Psi_1 + C_2\,a_2\,\Psi_2 + C_3\,a_3\,\Psi_3 + C_4\,a_4\,\Psi_4 + \ldots + C_n\,a_n\,\Psi_n \text{ ----(6)}$$

a_1 , a_2, a_3, a_4, a_n are Eigen values for Eigen functions Ψ_1, Ψ_2 , Ψ_3 , Ψ_4 Ψ_n respectively. By taking the conjugated complex for eq-1

$$\Psi^* = C^*_1\,\Psi^*_1 + C^*_2\,\Psi^*_2 + C^*_3\,\Psi^*_3 + C^*_4\,\Psi^*_4 + \ldots + C^*_n\,\Psi^*_n \text{ ------(7)}$$

By multiply the first side of eq-6 by the first side of eq-7 and the second side by the second side for the same equations.

$$\int \Psi^* \hat{A}\Psi \partial\tau = C^*_1 C_1 a_1 \int \Psi^*_1 \Psi_1 \partial\tau + C^*_2 C_2 a_2 \int \Psi^*_2 \Psi_2 \partial\tau +$$

$$C^*_3 C_3 a_3 \int \Psi^*_3 \Psi_3 \partial\tau + C^*_4 C_4 a_4 \int \Psi^*_4 \Psi_4 \partial\tau + . + C^*_n C_n a_n \int \Psi^*_n \Psi_n \partial\tau \text{-- } (8)$$

Eq-8 is clear from the terms of $C^*_i C_j a_j \int \Psi^*_i \Psi_j \partial\tau$ because its equal to zero (orthogonal functions).due to all terms in right side equal to one in equation -8 , so that :-

$$\int \Psi^* \hat{A}\Psi \partial\tau = C^*_1 C_1 a_1 + C^*_2 C_2 a_2 + C^*_3 C_3 a_3 + C^*_4 C_4 a_4 + \ldots + C^*_n C_n a_n \text{-} (9)$$

According to five postulate of quantum mechanics ,the right side of equation -9 represent the average value of unobservable quantity that's suitable to the mathematical operator when the system in the state of Ψ,by using the average value(ā)..can be write eq-9 as $a^- = \int \Psi^* \hat{A}\Psi \partial\tau$----------(10) by this equation can be find a lot of measuring for $a_1, a_2, \ldots$..ect for any physical measuring quantity. the average value equal to summation of measuring divided by the number of measuring .this equation is transfer from the pure mathematical quantum mechanics theory into real physical system.

Example2//

Prove that $C^*_1 C_1 + C^*_2 C_2 = 1$,,According to $\Psi = C_1 \Psi_1 + C_2 \Psi_2$ where C_1 ,and C_2 are constants of system case and the probability of a_1 value equal to $C^*_1 C_1$ and a_2 equal to $C^*_2 C_2$,since a1 and a2 are eigen values for functions Ψ_1 , Ψ_2 respectively.

Solving:-

Normalization condition is $\int \Psi^* \Psi \partial\tau = 1$,so that

$$\int \Psi^* \Psi \partial\tau = 1 = \int (C_1 \Psi_1 + C_2 \Psi_2)^* (C_1 \Psi_1 + C_2 \Psi_2) \partial\tau$$ therefore

$$C^*_1 C_1 \int \Psi^*_1 \Psi_1 \partial\tau + C^*_1 C_2 \int \Psi^*_1 \Psi_2 \partial\tau + C^*_2 C_1 \int \Psi^*_2 \Psi_1 \partial\tau + C^*_2 C_2 \int \Psi^*_2 \Psi_2 \partial\tau = 1$$

$\int \Psi^*_1 \Psi_2 \partial\tau = 0$, $\int \Psi^*_2 \Psi_1 \partial\tau = 0$ (orthogonal conditions)

and

$\int \Psi^*_1 \Psi_1 \partial\tau = 1$, $\int \Psi^*_2 \Psi_2 \partial\tau = 1$ (normalization condition)

so that $C^*_1 C_1 + C^*_2 C_2 = 1$