3.6.2 Bingham plastic fluids

The laminar axial flow of Bingham plastic fluids through a concentric annulus has generated even more interest than that for the power-law fluids, e.g. see refs. [Laird, 1957; Fredrickson and Bird, 1958; Bird *et al.*, 1983; Fordham *et al.*, 1991]. The main feature which distinguishes the flow of a Bingham plastic fluid from that of a power-law fluid is the existence of a plug region in which the shear stress is less than the yield stress. Figure 3.18 shows qualitatively the salient features of the velocity distribution in an annulus; the corresponding profile for a fluid without the yield stress (e.g. power-law fluid) is also shown for the sake of comparison.

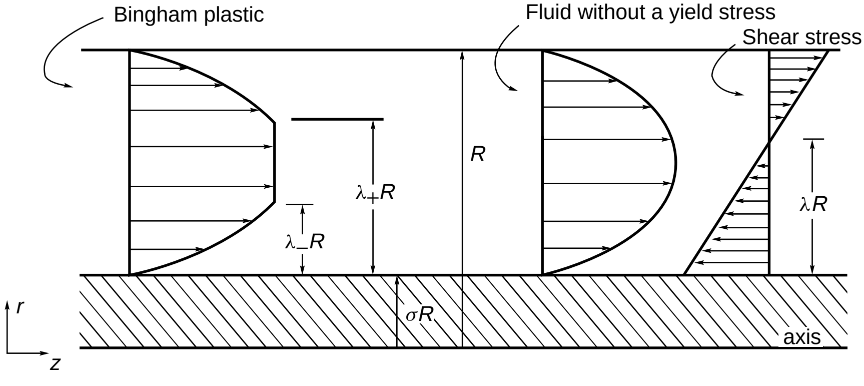

Figure 3.18 *Schematics of velocity profiles for Bingham plastic and power-law fluids in an concentric annulus*

In principle, the velocity distribution and the mean velocity of a Bingham plastic fluid flowing through an annulus can be deduced by substituting for the shear stress in equation (3.76) in terms of the Bingham plastic model, equation (3.10). However, the signs of the shear stress (considered positive in the same sense as the flow) and the velocity gradients in the two flow regions need to be treated with special care. With reference to the sketch shown in Figure 3.18, the shearing force on the fluid is positive ($\sigma R \le r \le \lambda_- R$) where the velocity gradient is also positive. Thus, in this region:

$$\tau_{rz} = \tau_0^B + \mu_B \left(\frac{\mathrm{d}V_z}{\mathrm{d}r}\right) \tag{3.83}$$

On the otherhand, in the region $\lambda_+ R \le r \le R$, the velocity gradient is negative and the shearing force is also in the negative r-direction and hence

$$-\tau_{rz} = \tau_0^B + \mu_B \left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right) \tag{3.84}$$

Equations (3.83) and (3.84) can now be substituted in equation (3.76) and integrated to deduce the velocity distributions. The constants of integration can be evaluated by using the no-slip boundary condition at both $r = \sigma R$ and $r = R$. However, the boundaries of the plug existing in the middle of the annulus are not yet known; nor is the plug velocity known. These unknowns are evaluated by applying the following three conditions, namely, the continuity of velocity at $r = \lambda_- R$ and $r = \lambda_+ R$, the velocity gradient is also zero at these boundaries and finally, the force balance on the plug of fluid:

$$2\pi R(\lambda_+ + \lambda_-)\tau_0^B = \left(\frac{-\Delta p}{L}\right. \cdot \pi((\lambda_+ R)^2 - (\lambda_- R)^2) \qquad (3.85)$$

Unfortunately, the algebraic steps required to carry out the necessary integrations and the evaluation of the constants are quite involved and tedious. Thus, these are not presented here and readers are referred to the original papers [Laird, 1957] or to the book by Skelland [1967] for detailed derivations. Instead consideration is given here to the practical problem of

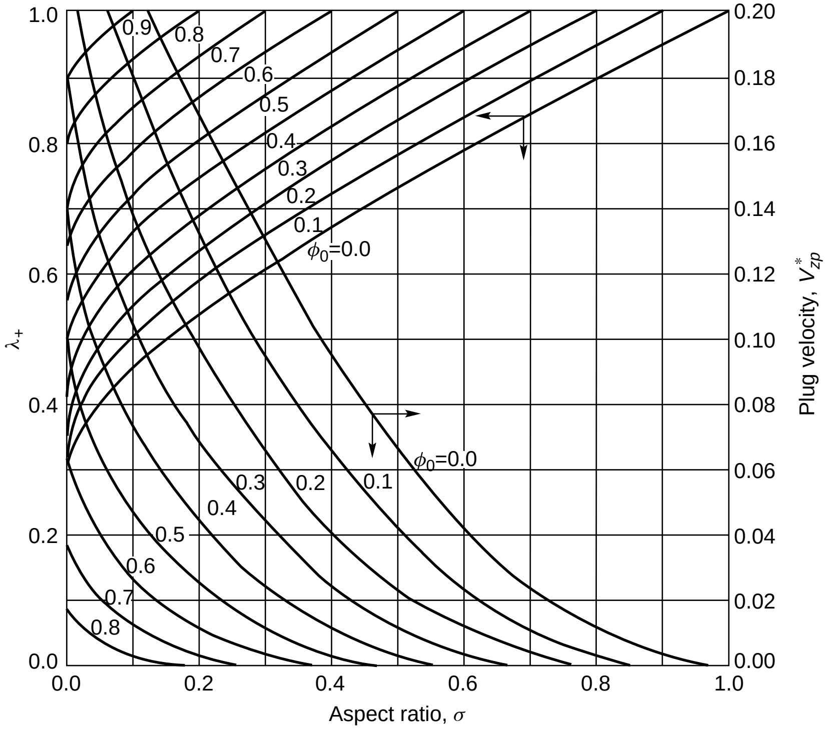

Figure 3.19 *Dimensionless plug velocity and plug size for laminar Bingham plastic flow in an annulus*

estimating the necessary pressure gradient to maintain a fixed flow rate of a Bingham plastic fluid or vice versa. Fredrickson and Bird [1958] organised their numerical solutions of the equations presented above in terms of the following dimensionless parameters:

$$\text{dimensionless velocity: } V_z^* = \frac{2\mu_B V_z}{R^2\left(\dfrac{-\Delta p}{L}\right)}$$

$$\text{dimensionless yield stress: } \phi_0 = \frac{2\tau_0^B}{R\left(\dfrac{-\Delta p}{L}\right)}$$

$$\text{dimensionless flowrate: } \Omega = \frac{Q}{Q_N}$$

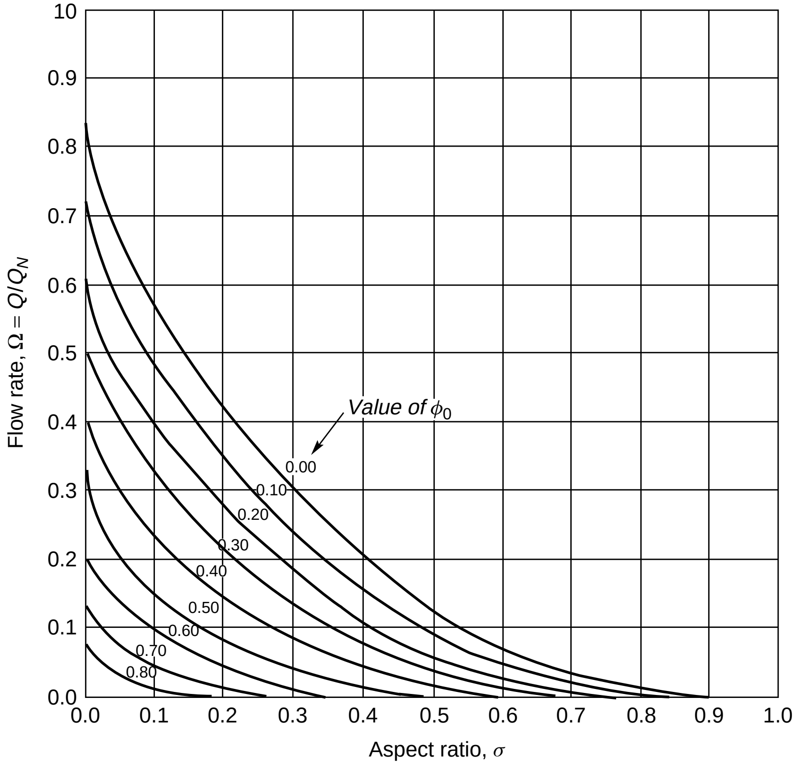

Figure 3.20 *Dimensionless flowrate for Bingham plastic fluids in laminar flow through an annulus*

where Q_N is the flow rate of a Newtonian liquid of viscosity, μ_B. Thus,

$$Q_N = \frac{\pi R^4}{8\mu_B}\left(\frac{-\Delta p}{L}\right.$$

Fredrickson and Bird [1958] presented three charts (Figures 3.19–3.21) showing relationships between V_z^*, ϕ_0, Ω, σ and λ_+.

For given values of the rheological constants (μ_B, τ_0^B), pressure gradient $(-\Delta p/L)$ and the dimensions of the annulus (σ, R), the values of λ_+ and the plug velocity V_{zp}^* can be read from Figure 3.19 and the value of Ω from Figure 3.20 from which the volumetric rate of flow, Q, can be estimated. For the reverse calculation, the group (Ω/ϕ_0) is independent of the pressure gradient and one must use Figure 3.21 to obtain the value of ϕ_0 and thus evaluate the required pressure gradient $(-\Delta p/L)$.

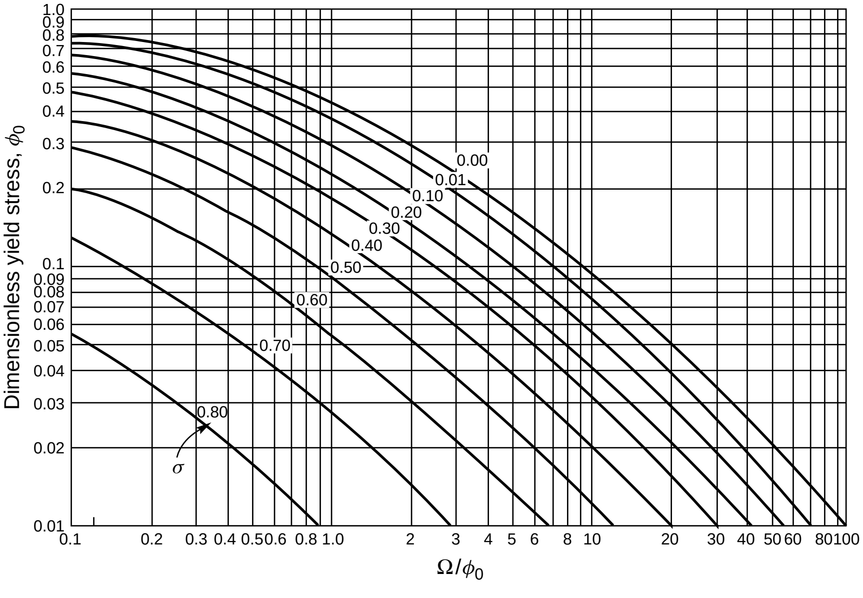

Figure 3.21 *Chart for the estimation of pressure gradient for laminar flow of Bingham plastic fluids in an annulus*

Example 3.11

A molten chocolate (density = 1500 kg/m^3) flows through a concentric annulus of inner and outer radii 10 mm and 20 mm, respectively, at 30°C at the constant flow rate of 0.03 m^3/min. The steady-shear behaviour of the chocolate can be approximated by a Bingham plastic model with $\tau_0^B = 35$ Pa and $\mu_B = 1$ Pa·s.

(a) Estimate the required pressure gradient to maintain the flow, and determine the velocity and the size of the plug.
(b) Owing to a pump malfunction, the available pressure gradient drops by 25% of the value calculated in (a), what will be the new flow rate?

Solution

Part (a):

In this case, $\tau_0^B = 35\,\text{Pa}$, $\mu_B = 1\,\text{Pa·s}$

$$Q = 0.03\ \text{m}^3/\text{min} = \frac{0.03}{60}\text{m}^3/\text{s}$$

$$\sigma = \frac{10}{20} = 0.5;\ R = 20 \times 10^{-3}\,\text{m}$$

Since the pressure gradient $(-\Delta p/L)$ is unknown, one must use Figure 3.21.

$$\therefore\ \frac{\Omega}{\phi_0} = \frac{4\mu_B Q}{\pi R^3 \tau_0^B} = \frac{4 \times 1 \times (0.03/60)}{3.14 \times (20 \times 10^{-3})^3 \times 35} = 2.28$$

For $\dfrac{\Omega}{\phi_0} = 2.28$ and $\sigma = 0.5$, Figure 3.21 gives:

$$\phi_0 = \ \sim 0.048$$

$$\therefore\ \left(\frac{-\Delta p}{L}\right) = \frac{2\tau_0^B}{R\phi_0} = \frac{2 \times 35}{20 \times 10^{-3} \times 0.048}$$

$$= 73\,000\,\text{Pa/m}$$

$$= 73\,\text{kPa/m}$$

Now from Figure 3.19, $\phi_0 = 0.048$ and $\sigma = 0.5$,

$$V_{z,p}^* = \sim 0.05$$

$$\lambda_+ = 0.76$$

From the definition of V_z^*, we have

$$V_{zp}^* = \frac{2\mu_B V_{zp}}{R^2\left(\dfrac{-\Delta p}{L}\right)}$$

$$0.05 = \frac{2 \times 1 \times V_{zp}}{(20 \times 10^{-3})^2 (73\,000)}$$

$$\therefore\ V_{zp} = 0.73\,\text{m/s}$$

i.e. the plug in the central region has a velocity of 0.73 m/s (compared with the mean velocity of $Q/\pi R^2(1 - \sigma^2)$, i.e. 0.53 m/s).

From equation (3.85):

$$(\lambda_+ - \lambda_-)\frac{R}{2}\left(\frac{-\Delta p}{L}\right) = \tau_0^B$$

Substitution of values gives $\lambda_- = 0.71$. Thus the plug region extends from $\lambda_- R$ to $\lambda_+ R$, i.e. from 14.2 to 15.2 mm. These calculations assume the flow to be laminar. As a first approximation, one can define the corresponding Reynolds number based on the hydraulic diameter, D_h.

$$D_h = \frac{4 \times \text{Flow area}}{\text{wetted perimeter}} = \frac{4\pi R^2(1-\sigma^2)}{2\pi R(1+\sigma)} = 2R(1-\sigma)$$

$$= 2 \times 20 \times 10^{-3}(1-0.5) = 0.02\,\text{m}$$

$$\text{Reynolds number, Re} = \frac{\rho V D_h}{\mu_B} = \frac{1500 \times 0.53 \times 0.02}{1} = 16$$

The flow is thus likely to be streamline.

Part (b): In this case, the available pressure gradient is only 75% of the value calculated above,

$$\frac{-\Delta p}{L} = 73 \times 0.75 = 54.75\,\text{kPa/m}$$

We can now evaluate ϕ_0:

$$\phi_0 = \frac{2\tau_0^B}{R\left(\frac{-\Delta p}{L}\right)} = \frac{2 \times 35}{20 \times 10^{-3} \times 54.75 \times 1000} = 0.064$$

From Figure 3.20, for $\phi_0 = 0.064$ and $\sigma = 0.5$,

$$\Omega = \frac{Q}{Q_N} = \sim 0.1$$

$$\therefore\ Q = 0.1 \times Q_N = 0.1 \times \frac{\pi R^4}{8\mu_B}\left(\frac{-\Delta p}{L}\right)$$

$$= \frac{0.1 \times 3.14 \times (20 \times 10^{-3})^4}{8 \times 1} \times 54.75 \times 1000\,\text{m}^3/\text{s}$$

$$= 0.000344\,\text{m}^3/\text{s or } 0.0206\,\text{m}^3/\text{min}$$

Two observations can be made here. The 25% reduction in the available pressure gradient has lowered the flow rate by 31%. Secondly, in this case the flow rate is only one tenth of that of a Newtonian fluid of the same viscosity as the plastic viscosity of the molten chocolate!

This section is concluded by noting that analogous treatments for the concentric and eccentric annular flow of Herschel–Bulkley and other viscosity models are also available in the literature [Hanks, 1979; Uner *et al*., 1988; Walton and Bittleston, 1991; Fordham *et al*., 1991; Gücüyener and Mehmeteoglu, 1992].