The Mean Value Theorem

Rolle's Theorem

THEOREM 3—Rolle's Theorem Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution We define the continuous function

$$f(x) = x^3 + 3x + 1.$$

Since $f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem tells us that the graph of f crosses the x-axis somewhere in the open interval $(-1, 0)$. (See Figure 1.) The derivative

$$f'(x) = 3x^2 + 3$$

is never zero (because it is always positive). Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero. Therefore, f has no more than one zero. ■

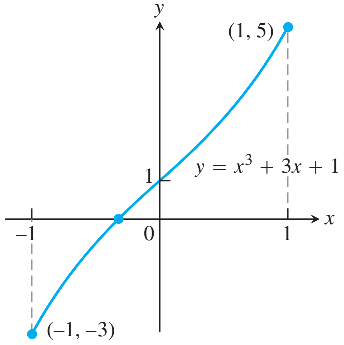

FIGURE 1 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0.

The Mean Value Theorem

THEOREM 4—The Mean Value Theorem Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

EXAMPLE 2 The function $f(x) = x^2$ (Figure 2) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists.

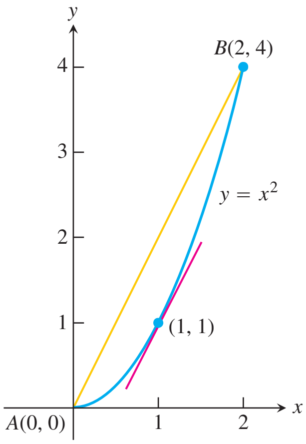

FIGURE 2 As we find in Example 2, is $c = 1$ where the tangent is parallel to the chord.

A Physical Interpretation

We can think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change. Then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

EXAMPLE 3 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. The Mean Value Theorem says that at some point during the acceleration the speedometer must read exactly 30 mph (44 ft/sec) (Figure 3).

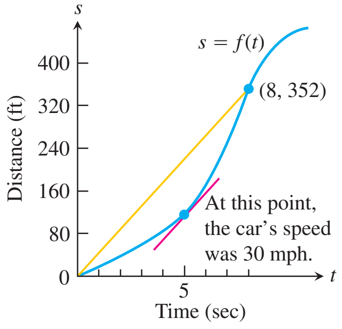

FIGURE 3 Distance versus elapsed time for the car in Example 3.

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

EXAMPLE 4 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

Solution Since the derivative of $g(x) = -\cos x$ is $g'(x) = \sin x$, we see that f and g have the same derivative. Corollary 2 then says that $f(x) = -\cos x + C$ for some constant C. Since the graph of f passes through the point $(0, 2)$, the value of C is determined from the condition that $f(0) = 2$:

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The function is $f(x) = -\cos x + 3$.

Exercises 4.2

Checking the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–8.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$

2. $f(x) = x^{2/3}, \quad [0, 1]$

3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$

4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$

5. $f(x) = \sin^{-1} x, \quad [-1, 1]$

7. $f(x) = x^3 - x^2, \quad [-1, 2]$

8. $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

Which of the functions in Exercises 9–14 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

9. $f(x) = x^{2/3}, \quad [-1, 8]$

10. $f(x) = x^{4/5}, \quad [0, 1]$

11. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$

12. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$

13. $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$

14. $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$

15. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

16. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Indeterminate Forms and L'Hôpital's Rule

We use $0/0$, $\infty \cdot 0$, $\infty - \infty$, 0^0, and 1^∞ as a notation for an expression known as an **indeterminate form**.

Indeterminate Form $0/0$

THEOREM 6— L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

EXAMPLE 1 The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $$\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = \left.\frac{3 - \cos x}{1}\right|_{x=0} = 2$$

(b) $$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1 + x}}}{1} = \frac{1}{2}$$

(c) $$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{(1/2)(1 + x)^{-1/2} - 1/2}{2x} \qquad \text{Still } \frac{0}{0}\text{; differentiate again.}$$

$$= \lim_{x \to 0} \frac{-(1/4)(1 + x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

(d) $$\lim_{x \to 0} \frac{x - \sin x}{x^3} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

EXAMPLE 2 Be careful to apply l'Hôpital's Rule correctly:

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0. \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Up to now the calculation is correct, but if we continue to differentiate in an attempt to apply l'Hôpital's Rule once more, we get

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2},$$

which is not the correct limit. L'Hôpital's Rule can only be applied to limits that give indeterminate forms, and $0/1$ is not an indeterminate form. ■

L'Hôpital's Rule applies to one-sided limits as well.

EXAMPLE 3 In this example the one-sided limits are different.

(a) $$\lim_{x \to 0^+} \frac{\sin x}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Positive for } x > 0$$

(b) $$\lim_{x \to 0^-} \frac{\sin x}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty \qquad \text{Negative for } x < 0$$ ■

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminant form like ∞/∞, $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. We first consider the form ∞/∞.

In more advanced treatments of calculus it is proved that l'Hôpital's Rule applies to the indeterminate form ∞/∞ as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x \to a$, a may be either finite or infinite. Moreover, $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

EXAMPLE 4 Find the limits of these $\infty\ \ \infty$ forms:

$$\lim_{x\to\pi/2}\frac{\sec x}{1+\tan x}$$

Solution

The numerator and denominator are discontinuous at $x=\pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x=\pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-}\frac{\sec x}{1+\tan x} \qquad \frac{\infty}{\infty}\text{ from the left}$$

$$=\lim_{x\to(\pi/2)^-}\frac{\sec x\tan x}{\sec^2 x}=\lim_{x\to(\pi/2)^-}\sin x=1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

EXAMPLE 5 Find the limits of these $\infty\cdot 0$ forms:

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$$

Solution

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)=\lim_{h\to0^+}\left(\frac{1}{h}\sin h\right)=\lim_{h\to0^+}\frac{\sin h}{h}=1 \qquad \infty\cdot 0;\text{ Let } h=1/x.$$

EXAMPLE 6 Find the limit of this $\infty-\infty$ form:

$$\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right).$$

Solution If $x\to0^+$, then $\sin x\to0^+$ and

$$\frac{1}{\sin x}-\frac{1}{x}\to\infty-\infty.$$

Similarly, if $x\to0^-$, then $\sin x\to0^-$ and

$$\frac{1}{\sin x}-\frac{1}{x}\to-\infty-(-\infty)=-\infty+\infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x} \qquad \text{Common denominator is } x\sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to0}\frac{x-\sin x}{x\sin x} \qquad \frac{0}{0}$$

$$=\lim_{x\to0}\frac{1-\cos x}{\sin x+x\cos x} \qquad \text{Still }\frac{0}{0}$$

$$=\lim_{x\to0}\frac{\sin x}{2\cos x-x\sin x}=\frac{0}{2}=0.$$

Exercises 4.5

Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied .

1. $\lim_{x\to-2} \dfrac{x+2}{x^2-4}$

2. $\lim_{x\to0} \dfrac{\sin 5x}{x}$

3. $\lim_{x\to\infty} \dfrac{5x^2-3x}{7x^2+1}$

4. $\lim_{x\to1} \dfrac{x^3-1}{4x^3-x-3}$

5. $\lim_{x\to0} \dfrac{1-\cos x}{x^2}$

6. $\lim_{x\to\infty} \dfrac{2x^2+3x}{x^3+x+1}$

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x\to2} \dfrac{x-2}{x^2-4}$

8. $\lim_{x\to-5} \dfrac{x^2-25}{x+5}$

9. $\lim_{t\to-3} \dfrac{t^3-4t+15}{t^2-t-12}$

10. $\lim_{t\to1} \dfrac{3t^3-3}{4t^3-t-3}$

11. $\lim_{x\to\infty} \dfrac{5x^3-2x}{7x^3+3}$

12. $\lim_{x\to\infty} \dfrac{x-8x^2}{12x^2+5x}$

13. $\lim_{t\to0} \dfrac{\sin t^2}{t}$

14. $\lim_{t\to0} \dfrac{\sin 5t}{2t}$

15. $\lim_{x\to0} \dfrac{8x^2}{\cos x-1}$

16. $\lim_{x\to0} \dfrac{\sin x-x}{x^3}$

17. $\lim_{\theta\to\pi/2} \dfrac{2\theta-\pi}{\cos(2\pi-\theta)}$

18. $\lim_{\theta\to-\pi/3} \dfrac{3\theta+\pi}{\sin(\theta+(\pi/3))}$

19. $\lim_{\theta\to\pi\ 2} \dfrac{1-\sin\theta}{1+\cos 2\theta}$

$\lim_{x\to1} \dfrac{x-1}{\ln x-\sin \pi x}$

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Monotonic Functions and the First Derivative Test

Increasing Functions and Decreasing Functions

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

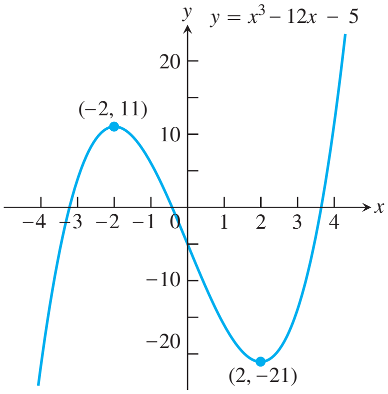


FIGURE 1 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

EXAMPLE 1 Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and on which f is decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$$

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f to create nonoverlapping open intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval. The behavior of f is determined by then applying Corollary 3 to each subinterval. The results are summarized in the following table, and the graph of f is given in Figure 4.20.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

EXAMPLE 2 Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

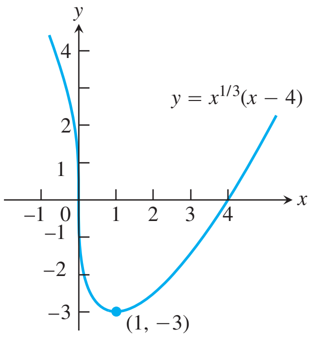


FIGURE 2 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}\left(x - 1\right) = \frac{4(x - 1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x-axis into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points, as summarized in the following table.

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum since f is decreasing on $(-\infty, 1]$ and increasing on $[1, \infty)$. Figure 2 shows this value in relation to the function's graph. ■

Exercises 4.3

Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

a. What are the critical points of f?

b. On what intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x - 1)$

2. $f'(x) = (x - 1)(x + 2)$

3. $f'(x) = (x - 1)^2(x + 2)$

4. $f'(x) = (x - 1)^2(x + 2)^2$

6. $f'(x) = (x - 7)(x + 1)(x + 5)$

7. $f'(x) = \dfrac{x^2(x - 1)}{x + 2}, \quad x \neq -2$

8. $f'(x) = \dfrac{(x - 2)(x + 4)}{(x + 1)(x - 3)}, \quad x \neq -1, 3$

9. $f'(x) = 1 - \dfrac{4}{x^2}, \quad x \neq 0$

10. $f'(x) = 3 - \dfrac{6}{\sqrt{x}}, \quad x \neq 0$

11. $f'(x) = x^{-1/3}(x + 2)$

12. $f'(x) = x^{-1/2}(x - 3)$

13. $f'(x) = (\sin x - 1)(2\cos x + 1),\ 0 \le x \le 2\pi$

14. $f'(x) = (\sin x + \cos x)(\sin x - \cos x),\ 0 \le x \le 2\pi$

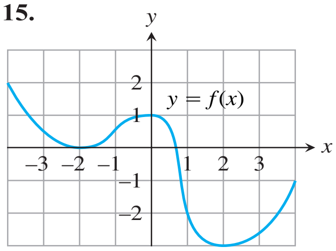
Identifying Extrema

In Exercises 15–44:

a. Find the open intervals on which the function is increasing and decreasing.

b. Identify the function's local and absolute extreme values, if any, saying where they occur.

15.



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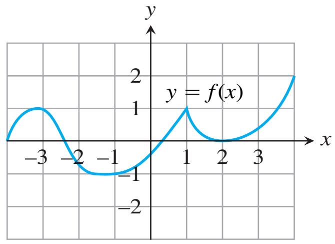

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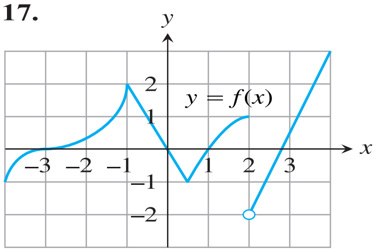

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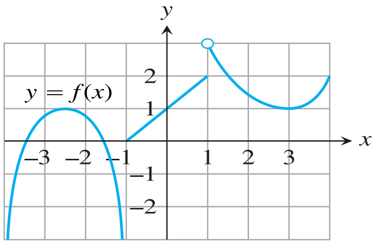


19. $g(t) = -t^2 - 3t + 3$

20. $g(t) = -3t^2 + 9t + 5$

21. $h(x) = -x^3 + 2x^2$

22. $h(x) = 2x^3 - 18x$

23. $f(\theta) = 3\theta^2 - 4\theta^3$

24. $f(\theta) = 6\theta - \theta^3$

25. $f(r) = 3r^3 + 16r$

26. $h(r) = (r + 7)^3$

27. $f(x) = x^4 - 8x^2 + 16$

28. $g(x) = x^4 - 4x^3 + 4x^2$

29. $H(t) = \frac{3}{2}t^4 - t^6$

30. $K(t) = 15t^3 - t^5$

31. $f(x) = x - 6\sqrt{x - 1}$

32. $g(x) = 4\sqrt{x} - x^2 + 3$

33. $g(x) = x\sqrt{8 - x^2}$

34. $g(x) = x^2\sqrt{5 - x}$

35. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$

36. $f(x) = \dfrac{x^3}{3x^2 + 1}$

37. $f(x) = x^{1/3}(x + 8)$

38. $g(x) = x^{2/3}(x + 5)$

40. $k(x) = x^{2/3}(x^2 - 4)$

57. $f(x) = \sin 2x, \quad 0 \leq x \leq \pi$

58. $f(x) = \sin x - \cos x, \quad 0 \leq x \leq 2\pi$

59. $f(x) = \sqrt{3}\cos x + \sin x, \quad 0 \leq x \leq 2\pi$

60. $f(x) = -2x + \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

61. $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \leq x \leq 2\pi$

62. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \leq x \leq \pi$

63. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

64. $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

Concavity and Curve Sketching

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.

2. If $f'' < 0$ on I, the graph of f over I is concave down.

If $y = f(x)$ is twice-differentiable, we will use the notations f'' and y'' interchangeably when denoting the second derivative.

EXAMPLE 1

(a) The curve $y = x^3$ is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 1) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ∎

EXAMPLE 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 2). ∎

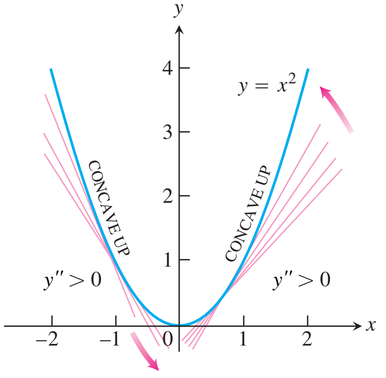


FIGURE 1 The graph of $f(x) = x^2$ is concave up on every interval

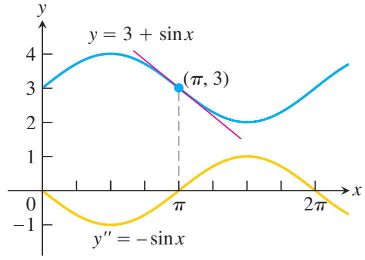


FIGURE 2 Using the sign of to y'' determine the concavity of y

Points of Inflection

DEFINITION A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

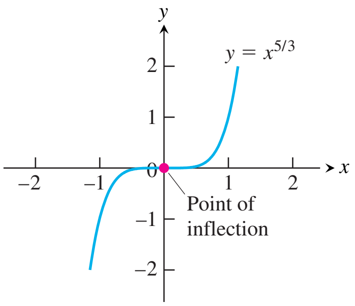


FIGURE 3 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$

EXAMPLE 3 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 3. ■

Here is an example showing that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

EXAMPLE 4 The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. ■

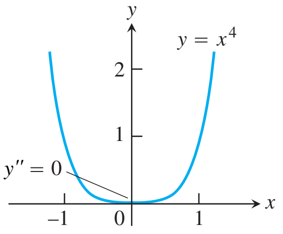


FIGURE 4 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$

EXAMPLE 5 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 5. ■

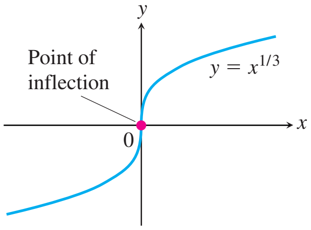


FIGURE 5 A point of inflection where y' and y'' fail to exist .

EXAMPLE 6 A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t-1)(3t-11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left

Notice that the first derivative $(v = s')$ is zero at the critical points $t = 1$ and $t = 11/3$.

Interval	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

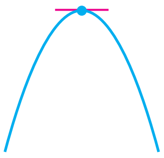
The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest) at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

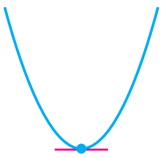
Interval	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

The particle starts out moving to the right while slowing down, and then reverses and begins moving to the left at $t = 1$ under the influence of the leftward acceleration over the time interval $[0, 7/3)$. The acceleration then changes direction at $t = 7/3$ but the particle continues moving leftward, while slowing down under the rightward acceleration. At $t = 11/3$ the particle reverses direction again: moving to the right in the same direction as the acceleration.

Second Derivative Test for Local Extrema



$f' = 0, f'' < 0$ $\Rightarrow$ local max



$f' = 0, f'' > 0$ $\Rightarrow$ local min

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

EXAMPLE 7 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where f is concave up or concave down.

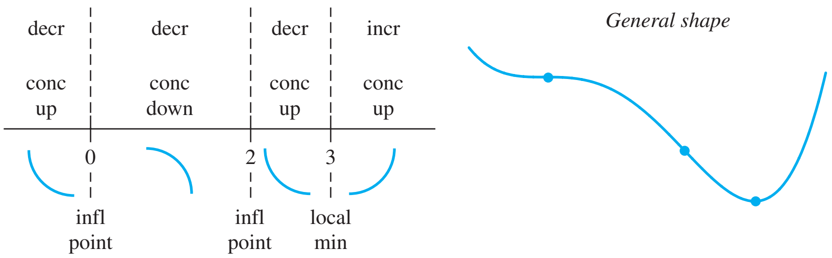
Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

(d) Summarizing the information in the last two tables, we obtain the following.

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

The general shape of the curve is shown in the accompanying figure.



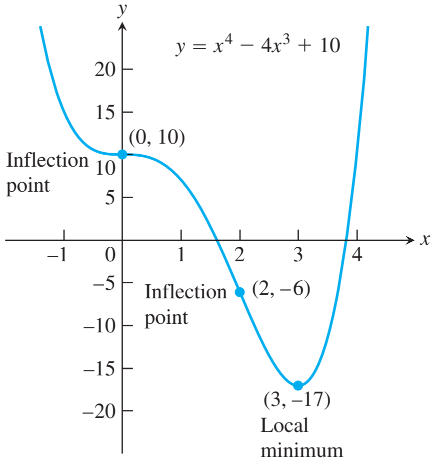


FIGURE 6 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 7).

(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 6 shows the graph of f. ■

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist (see Section 2.6).
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

EXAMPLE 8 Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x-intercept at $x = -1$, y-intercept ($y = 1$) at $x = 0$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2\cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2\cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

After some algebra

3. *Behavior at critical points*. The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$ yielding a relative maximum by the Second Derivative test.
4. *Increasing and decreasing*. We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

5. *Inflection points*. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.

6. *Asymptotes*. Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} \qquad \text{Expanding numerator}$$

$$= \frac{1+(2/x)+(1/x^2)}{(1/x^2)+1}. \qquad \text{Dividing by } x^2$$

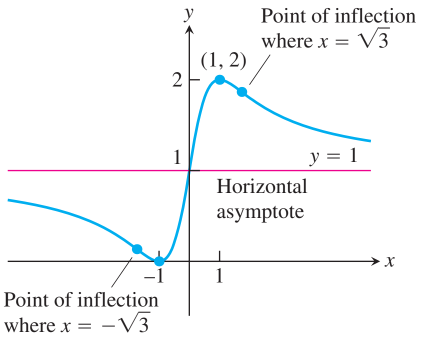


FIGURE 7 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \le y \le 2$).

7. The graph of f is sketched in Figure 7. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■

EXAMPLE 9 Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$.

Solution

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.

2. We calculate the derivatives of the function, but first rewrite it in order to simplify our computations:

$$f(x) = \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x} \qquad \text{Function simplified for differentiation}$$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2} \qquad \text{Combine fractions to solve easily } f'(x) = 0.$$

$$f''(x) = \frac{4}{x^3} \qquad \text{Exists throughout the entire domain of } f$$

3. The critical points occur at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.

4. On the interval $(-\infty, -2)$ the derivative f' is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.

5. There are no points of inflection because $f''(x) < 0$ whenever $x < 0$, $f''(x) > 0$ whenever $x > 0$, and f'' exists everywhere and is never zero throughout the domain of f. The graph is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.

6. From the rewritten formula for $f(x)$, we see that

$$\lim_{x\to 0^+}\left(\frac{x}{2}+\frac{2}{x}\right) = +\infty \quad \text{and} \quad \lim_{x\to 0^-}\left(\frac{x}{2}+\frac{2}{x}\right) = -\infty,$$

so the y-axis is a vertical asymptote. Also, as $x \to \infty$ or as $x \to -\infty$, the graph of $f(x)$ approaches the line $y = x/2$. Thus $y = x/2$ is an oblique asymptote. ■

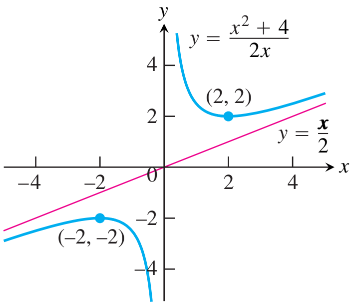


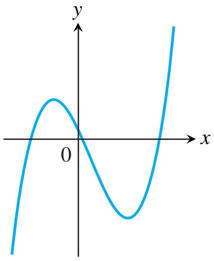
FIGURE 8 The graph of $y = \dfrac{x^2 + 4}{2x}$

Exercises 4.4

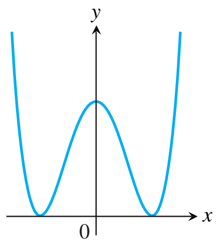
Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

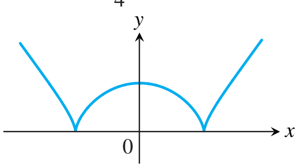
1. $y = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + \dfrac{1}{3}$



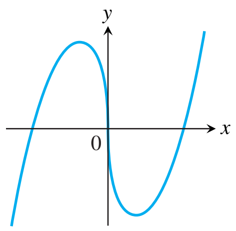
2. $y = \dfrac{x^4}{4} - 2x^2 + 4$



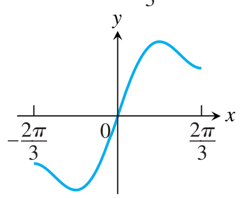
3. $y = \frac{3}{4}(x^2 - 1)^{2/3}$



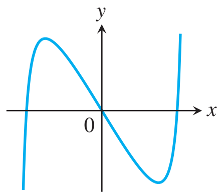
4. $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



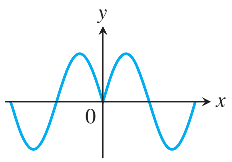
5. $y = x + \sin 2x, -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



6. $y = \tan x - 4x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

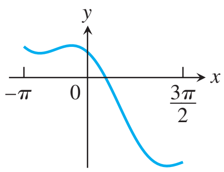


7. $y = \sin |x|, -2\pi \le x \le 2\pi$



NOT TO SCALE

8. $y = 2\cos x - \sqrt{2}x, -\pi \le x \le \frac{3\pi}{2}$



Graphing Equations

Use the steps of the graphing procedure on page 248 to graph the equations in Exercises 9–58. Include the coordinates of any local and absolute extreme points and inflection points.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$

20. $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$

22. $y = x\left(\dfrac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$

26. $y = \dfrac{4}{3}x - \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

27. $y = \sin x \cos x, \quad 0 \le x \le \pi$

28. $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

29. $y = x^{1/5}$

30. $y = x^{2/5}$

31. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

32. $y = \dfrac{\sqrt{1 - x^2}}{2x + 1}$

33. $y = 2x - 3x^{2/3}$

34. $y = 5x^{2/5} - 2x$

35. $y = x^{2/3}\left(\dfrac{5}{2} - x\right)$

36. $y = x^{2/3}(x - 5)$

37. $y = x\sqrt{8 - x^2}$

38. $y = (2 - x^2)^{3/2}$

39. $y = \sqrt{16 - x^2}$

40. $y = x^2 + \dfrac{2}{x}$

41. $y = \dfrac{x^2 - 3}{x - 2}$

42. $y = \sqrt[3]{x^3 + 1}$

43. $y = \dfrac{8x}{x^2 + 4}$

44. $y = \dfrac{5}{x^4 + 5}$

45. $y = |x^2 - 1|$

46. $y = |x^2 - 2x|$

Sketching the General Shape, Knowing y'

Each of Exercises 59–80 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use steps 2–4 of the graphing procedure on page 248 to sketch the general shape of the graph of f.

59. $y' = 2 + x - x^2$

60. $y' = x^2 - x - 6$

61. $y' = x(x - 3)^2$

62. $y' = x^2(2 - x)$

63. $y' = x(x^2 - 12)$

64. $y' = (x - 1)^2(2x + 3)$

91. $y = -\dfrac{x^2 - 2}{x^2 - 1}$

92. $y = \dfrac{x^2 - 4}{x^2 - 2}$

93. $y = \dfrac{x^2}{x + 1}$

94. $y = -\dfrac{x^2 - 4}{x + 1}$

95. $y = \dfrac{x^2 - x + 1}{x - 1}$

96. $y = -\dfrac{x^2 - x + 1}{x - 1}$

97. $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$

98. $y = \dfrac{x^3 + x - 2}{x - x^2}$

99. $y = \dfrac{x}{x^2 - 1}$

100. $y = \dfrac{x - 1}{x^2(x - 2)}$

65. $y' = (8x - 5x^2)(4 - x)^2$

66. $y' = (x^2 - 2x)(x - 5)^2$

67. $y' = \sec^2 x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

68. $y' = \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

69. $y' = \cot\dfrac{\theta}{2}, \quad 0 < \theta < 2\pi$

70. $y' = \csc^2\dfrac{\theta}{2}, \quad 0 < \theta < 2\pi$

71. $y' = \tan^2\theta - 1, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

72. $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

73. $y' = \cos t, \quad 0 \le t \le 2\pi$

74. $y' = \sin t, \quad 0 \le t \le 2\pi$

75. $y' = (x + 1)^{-2/3}$

76. $y' = (x - 2)^{-1/3}$

77. $y' = x^{-2/3}(x - 1)$

78. $y' = x^{-4/5}(x + 1)$

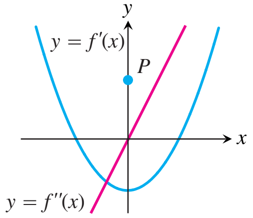
79. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

80. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

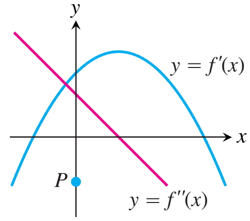
Sketching y from Graphs of y' and y''

Each of Exercises 81–84 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

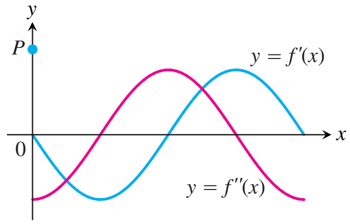
81.



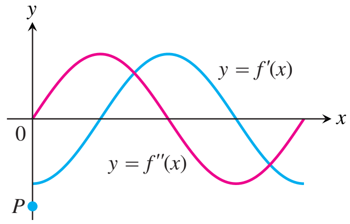
82.



83.



84.



Graphing Rational Functions

Graph the rational functions in Exercises 85–102.

85. $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$

86. $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

87. $y = \dfrac{x^4 + 1}{x^2}$

88. $y = \dfrac{x^2 - 4}{2x}$

89. $y = \dfrac{1}{x^2 - 1}$

90. $y = \dfrac{x^2}{x^2 - 1}$