Matrix equation

The vector equation is equivalent to a matrix equation of the form

$$A\mathbf{x} = \mathbf{b}$$

where *A* is an $m \times n$ matrix, **x** is a column vector with *n* entries, and **b** is a column vector with *m* entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The number of vectors in a basis for the span is now expressed as the *rank* of the matrix.

Row reduction

In **row reduction**, the linear system is represented as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right].$$

This matrix is then modified using elementary row operations until it reaches reduced row echelon form. There are three types of elementary row operations:

Type 1: Swap the positions of two rows.
Type 2: Multiply a row by a nonzero scalar.
Type 3: Add to one row a scalar multiple of another.

Because these operations are reversible, the augmented matrix produced always represents a linear system that is equivalent to the original.

There are several specific algorithms to row-reduce an augmented matrix, the simplest of which are Gaussian elimination and Gauss-Jordan elimination. The following computation shows Gauss-Jordan elimination applied to the matrix above:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

The last matrix is in reduced row echelon form, and represents the system $x = -15$, $y = 8$, $z = 2$. A comparison with the example in the previous section on the algebraic elimination of variables shows that these two methods are in fact the same; the difference lies in how the computations are written down.

Cramer's Rule

Recall the general 3×4 matrix used to solve systems of three equations:

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

This matrix will be used to solve systems by Cramer's Rule. We divide it into four separate 3×3 matrices:

$$D = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$D_x = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}$$

$$D_y = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}$$

$$D_z = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$$

D is the 3×3 coefficient matrix, and D_x, D_y, and D_z are each the result of substituting the constant column for one of the coefficient columns in D.

Cramer's Rule states that:

$x = \frac{detD_x}{detD}$
$y = \frac{detD_y}{detD}$
$z = \frac{detD_z}{detD}$

Thus, to solve a system of three equations with three variables using Cramer's Rule,

1. Arrange the system in the following form:

   $a_1x + b_1y + c_1z = d_1$
   $a_2x + b_2y + c_2z = d_2$
   $a_3x + b_3y + c_3z = d_3$

2. Create D, D_x, D_y, and D_z.
3. Find $detD$, $detD_x$, $detD_y$, and $detD_z$.
4. $x = \frac{detD_x}{detD}$, $y = \frac{detD_y}{detD}$, and $z = \frac{detD_z}{detD}$.

Note: If $detD = 0$ and $detD_x$, $detD_y$, or $detD_z \neq 0$, the system is inconsistent. If $detD = 0$ and $detD_x = detD_y = detD_z = 0$, the system has multiple solutions.

Example:Solve the following system:

$8x + 10z = 7y + 15$
$2x + 3y + 8z = 7$
$5y + 9 = 4x + 2z$

1. Rearrange the system:

   $8x - 7y + 10z = 15$
   $2x + 3y + 8z = 7$
   $-4x + 5y - 2z = -9$

2. Create the matrices:

$$D = \begin{bmatrix} 8 & -7 & 10 \\ 2 & 3 & 8 \\ -4 & 5 & -2 \end{bmatrix}$$

3.

4.

$$D_x = \begin{bmatrix} 15 & -7 & 10 \\ 7 & 3 & 8 \\ -9 & 5 & -2 \end{bmatrix}$$

5.

$$D_y = \begin{bmatrix} 8 & 15 & 10 \\ 2 & 7 & 8 \\ -4 & -9 & -2 \end{bmatrix}$$

6.

$$D_z = \begin{bmatrix} 8 & -7 & 15 \\ 2 & 3 & 7 \\ -4 & 5 & -9 \end{bmatrix}$$

7.

8. Find the determinants:

   $detD = (-48 + 224 + 100) - (-120 + 320 + 28) = 276 - 228 = 48$
   $detD_x = (-90 + 504 + 350) - (-270 + 600 + 98) = 764 - 428 = 336$
   $detD_y = (-112 - 480 - 180) - (-280 - 576 - 60) = -772 - (-916) = 144$
   $detD_z = (-216 + 196 + 150) - (-180 + 280 + 126) = 130 - 226 = -96$

9. $x = \frac{336}{48} = 7$. $y = \frac{144}{48} = 3$. $z = \frac{-96}{48} = -2$.

Thus, $(x, y, z) = (7, 3, -2)$.

Cramer's rule is an explicit formula for the solution of a system of linear equations, with each variable given by a quotient of two determinants. For example, the solution to the system

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

is given by

$$x = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 7 & 5 & 6 \\ 8 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 1 & 5 & -2 \\ 3 & 7 & 6 \\ 2 & 8 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 2 & 4 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}.$$

For each variable, the denominator is the determinant of the matrix of coefficients, while the numerator is the determinant of a matrix in which one column has been replaced by the vector of constant terms.

Though Cramer's rule is important theoretically, it has little practical value for large matrices, since the computation of large determinants is somewhat cumbersome. (Indeed, large determinants are most easily computed using row reduction.) Further, Cramer's rule has very poor numerical properties, making it unsuitable for solving even small systems reliably, unless the operations are performed in rational arithmetic with unbounded precision.

Matrix solution

If the equation system is expressed in the matrix form $A\mathbf{x} = \mathbf{b}$, the entire solution set can also be expressed in matrix form. If the matrix *A* is square (has *m* rows and *n*=*m* columns) and has full rank (all *m* rows are independent), then the system has a unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}$$

where A^{-1} is the inverse of *A*. More generally, regardless of whether *m*=*n* or not and regardless of the rank of *A*, all solutions (if any exist) are given using the Moore-Penrose pseudoinverse of *A*, denoted A^g, as follows:

$$\mathbf{x} = A^g\mathbf{b} + (I - A^gA)\mathbf{w}$$

where $\mathbf{w}$ is a vector of free parameters that ranges over all possible $n\times1$ vectors. A necessary and sufficient condition for any solution(s) to exist is that the potential solution obtained using $\mathbf{w} = \mathbf{0}$ satisfy $A\mathbf{x} = \mathbf{b}$— that is, that $AA^g\mathbf{b} = \mathbf{b}$. If this condition does not hold, the equation system is inconsistent and has no solution. If the condition holds, the system is consistent and at least one solution exists. For example, in the above-mentioned case in which *A* is square and of full rank, A^g simply equals A^{-1} and the general solution equation simplifies to $\mathbf{x} = A^{-1}\mathbf{b} + (I - A^{-1}A)\mathbf{w} = A^{-1}\mathbf{b} + (I - I)\mathbf{w} = A^{-1}\mathbf{b}$ as previously stated, where $\mathbf{w}$ has completely dropped out of the solution, leaving only a single solution. In other cases, though, $\mathbf{w}$ remains and hence an infinitude of potential values of the free parameter vector $\mathbf{w}$ give an infinitude of solutions of the equation.