

Week 3: Simplex Method I

1. Introduction

The simplex method computations are particularly tedious and repetitive. It attempts to move from one corner point of the solution space to a better corner point until the optimum is found. The development of the simplex method computations is facilitated by imposing two requirements on the constraints of the problem:

1-all constraints (with the exception of the nonnegative of the variable) are equations with nonnegative right-hand side.

2-all the variables are nonnegative.

1.1 convert inequalities into equation with nonnegative right-hand side

In (\leq) constraint, the right-hand side can be thought of as representing the limit on the availability of a resource, in which case the left-hand side would represent the usage of this limited resource by the activities (variables) of the model. The difference between the right-hand side and left-hand side of the (\leq) constraint thus yields the unused or **slack** amount of the resource.

To convert a (\leq) inequality to an equation, a nonnegative slack variable is **added** to the left-hand side of the constraint.

Example:

$$6x_1 + 4x_2 \leq 24$$

$$6x_1 + 4x_2 + s_1 = 24$$

$$s_1 \geq 0$$

a (\geq) constraint sets a lower limit on the activities of the LP model, so that the amount by which the left-hand side exceeds the minimum limit represents a **surplus**. The conversion from (\geq) to ($=$) is achieved by **subtracting** a nonnegative surplus variable from the left-hand side of the inequality.

Example:

$$x_1 + x_2 \geq 800$$

$$x_1 + x_2 - s_1 = 800$$

$$s_1 \geq 0$$

Example:

$$-x_1 + x_2 \leq -3$$

$$-x_1 + x_2 + s_1 = -3$$

Multiplying both sides by (-1) will render a nonnegative right-hand side

$$x_1 - x_2 - s_1 = 3$$

in the graphical method, the solution space is delineated by the half-space representing the constraints, and in the simplex method the solution space is represented by m simultaneous linear equations and n nonnegative variables.

The graphical solution space has an infinite number of solution points, but how can we draw a similar conclusion from the algebraic representation of the solution space.

The algebraic representation the number of equation (m) is always less than or equal to the number of variable (n), if $m=n$, and the equations are consistent, the system has only one solution, but if $m < n$, then the system of equations, again if consistent, will yield an infinite number of solutions.

In set of ($m \times n$) equations ($m < n$), if $n-m$ variables equal to zero and then solve the m equations for the remaining m variable, the resulting solution, if unique, is called a basic solution and must correspond to a (feasible or infeasible) corner point of the solution space. The maximum number of corner point is

$$C_m^n = \frac{n!}{m!(n-m)!}$$

Example:

Consider the following LP with two variables:

$$\text{Maximize } z = 2x_1 + 3x_2$$

Subject to:

$$2x_1 + x_2 \leq 4$$

$$x_1 + 2x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

convert to equality equation (constraint)

$$2x_1 + x_2 + s_1 = 4$$

$$x_1 + 2x_2 + s_2 = 5$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The system has $m = 2$ equations and $n = 4$ variables. Thus, according to the given rule, the corner

points can be determined algebraically by setting $n - m = 4 - 2 = 2$ variables equal to zero and then solving for the remaining $m = 2$ variables. For example, if we set $x_1 = 0$ and $x_2 = 0$, the equations provide the unique (basic) solution

$$s_1 = 4$$

$$s_2 = 5$$

This solution corresponds to point A in Figure below (convince yourself that $s_1 = 4$ and $s_2 = 5$ at point A). Another point can be determined by setting $s_1 = 0$ and $s_2 = 0$ and then solving the two equations

$$2x_1 + x_2 = 4$$

$$x_1 + 2x_2 = 5$$

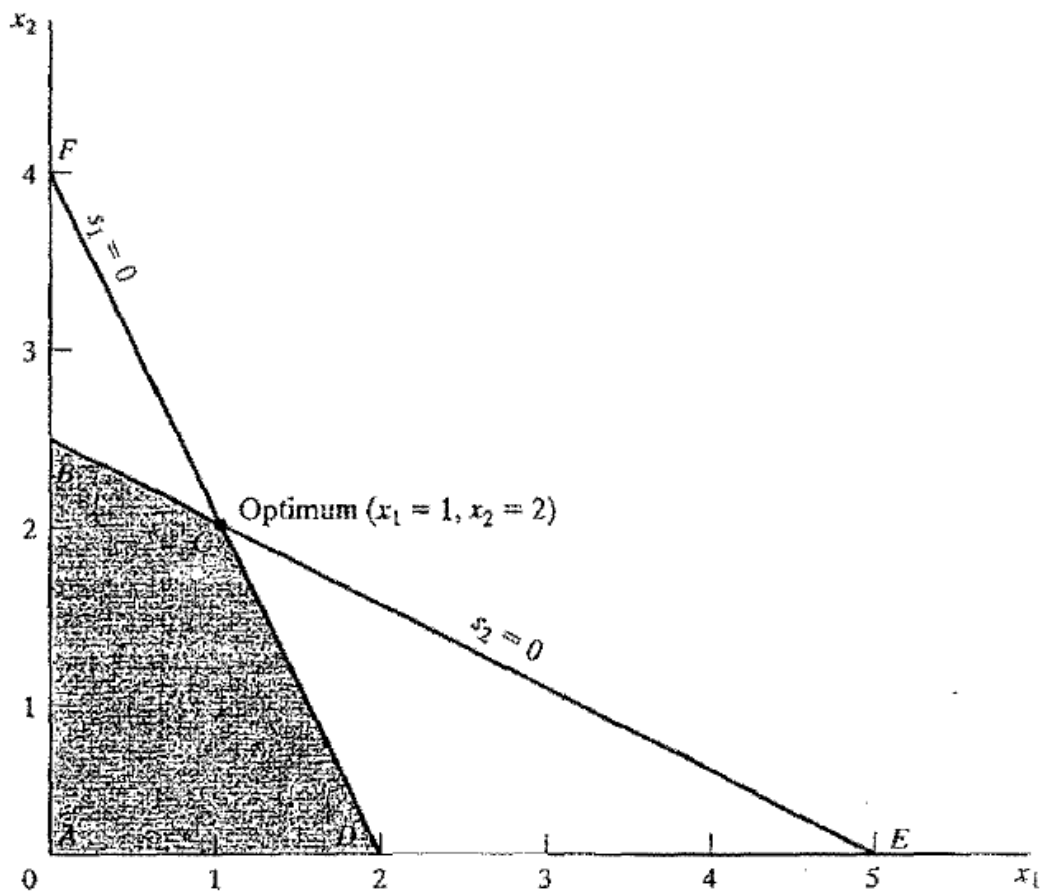
This yields the basic solution ($x_1 = 1, x_2 = 2$), which is point C

You probably are wondering how one can decide which $n - m$ variables should be set equal to zero to target a specific corner point. Without the benefit of the graphical solution (which is available only for two or three variables), we cannot say which ($n - m$) zero variables are associated with which corner point. But that does not prevent us from enumerating *all* the corner points of the solution space. Simply consider *all* combinations in which $n - m$ variables are set to zero and solve the resulting equations. Once done, the optimum solution is the feasible basic solution (corner point) that yields the best objective value.

In the present example we have $C = 6$ corner points. Looking at Figure below, we can immediately spot the four corner points $A, B, C,$ and D . Where, then, are the remaining two? In fact, points E and F also are corner points for the problem, but they are *infeasible* because they do not satisfy all the constraints. These infeasible corner points are not candidates for the optimum.

To summarize the transition from the graphical to the algebraic solution, the zero $n - m$ variables are known as **nonbasic variables**. The remaining m variables are called **basic variables** and their solution (obtained by solving the m equations) is referred to as **basic solution**. The following table provides all the basic and non basic solutions of the current example.

Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	feasible	Objective value, z
(x1,x2)	(s1,s2)	(4,5)	A	Yes	0
(x1,s1)	(x2,s2)	(4,-3)	F	No	-
(x1,s2)	(x2,s1)	(2.5,1.5)	B	Yes	7.5
(x2,s1)	(x1,s2)	(2,3)	D	Yes	4
(x2,s2)	(x1,s1)	(5,-6)	E	No	-
(s1,s2)	(x1,x2)	(1,2)	C	yes	8 (optimum)



2. simplex method

Rather than enumerating *all* the basic solutions (corner points) of the LP problem (example above), the simplex method investigates only a "select few" of these solutions. the simplex method starts at the origin (point A) where $x_1=0$ and $x_2=0$. At this starting point, the value of the objective function, z , is zero, and the logical question is whether an increase in nonbasic x_1 and/or x_2 above their current zero values can improve (increase) the value of z . We answer this question by investigating the objective function:

$$\text{Maximize } z=2x_1+3x_2$$

The function shows that an increase in either x_1 or x_2 (or both) above their current zero values will *improve* the value of z . The design of the simplex method calls for increasing *one variable at a time*, with the selected variable being the one with the *largest rate of improvement* in z . In the present example, the value of z will increase by 2 for each unit increase in x_1 and by 3 for each unit increase in x_2 . This means that the *rate* of improvement in the value of z is 2 for x_1 and 3 for x_2 . We thus elect to increase x_2 , the variable with the largest rate of improvement. Figure below shows that the value of x_2 must be increased until corner point B is reached (recall that stopping short of reaching corner point B is not optimal because a candidate for the optimum must be a corner point). At point B , the simplex method will then increase the value of x_1 to reach the improved corner point C , which is the optimum. The path of the simplex algorithm is thus defined as $A \rightarrow B \rightarrow C$. Each corner point along the path is associated with an iteration. It is important to note that the simplex method moves alongside the edges of the solution space, which means that the method cannot cut across the solution space, going from A to C directly.

We need to make the transition from the graphical solution to the algebraic solution by showing how the points A , B , and C are represented by their basic and nonbasic variables. The following table summarizes these representations:

Corner point	Basic variable	Nonbasic (zero) variable
A	s_1, s_2	x_1, x_2
B	s_1, x_2	x_1, s_2
C	x_1, x_2	s_1, s_2

Notice the change pattern in the basic and nonbasic variables as the solution moves along the path $A \rightarrow B \rightarrow C$. From A to B , nonbasic x_2 at A becomes basic at B and basic s_2 at A becomes nonbasic at B . In the terminology of the simplex method, we say that x_2 is the entering variable (because it enters the basic solution) and s_2 is the leaving variable (because it leaves the basic solution). In a similar manner, at point B , x_1 enters (the basic solution) and s_1 leaves, thus leading to point C .

Example:

We use the Reddy Mikks model to explain the details of the simplex method.

The problem is expressed in equation form as:

$$\text{Maximize } z=5x_1+4x_2+0s_1+0s_2+0s_3+0s_4$$

Subject to:

$$\begin{array}{rcl} 6x_1+4x_2+ s_1 & & =24 \\ x_1+2x_2 & +s_2 & =6 \\ -x_1+x_2 & & +s_3 =1 \end{array}$$

$$x_2 + s_4 = 2$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

The variables $s_1, s_2, s_3,$ and s_4 are the slacks associated with the respective constraints.

Next, we write the objective equation as

$$z - 5x_1 - 4x_2 = 0$$

In this manner, the starting simplex tableau can be represented as follows:

Basic	Z	X1	X2	S1	S2	S3	S4	Solution	
z	1	-5	-4	0	0	0	0	0	z-row
S1	0	6	4	1	0	0	0	26	S1-row
S2	0	1	2	0	1	0	0	6	S2-row
S3	0	-1	1	0	0	1	0	1	S3-row
S4	0	0	1	0	0	0	1	2	

solution associated with the starting iteration. The simplex iterations start at the origin $(x_1, x_2) = (0, 0)$ whose associated set of nonbasic and basic variables are defined as

$$\text{nonbasic (zero) variables: } (x_1, x_2)$$

$$\text{basic variables: } (s_1, s_2, s_3, s_4)$$

Substituting the nonbasic variables $(x_1, x_2) = (0, 0)$ and noting the special 0-1 arrangement of the coefficients of z and the basic variables (s_1, s_2, s_3, s_4) in the tableau, the following solution is immediately available (without any calculations):

$$z = 0$$

$$s_1 = 24$$

$$s_2 = 6$$

$$s_3 = 1$$

$$s_4 = 2$$

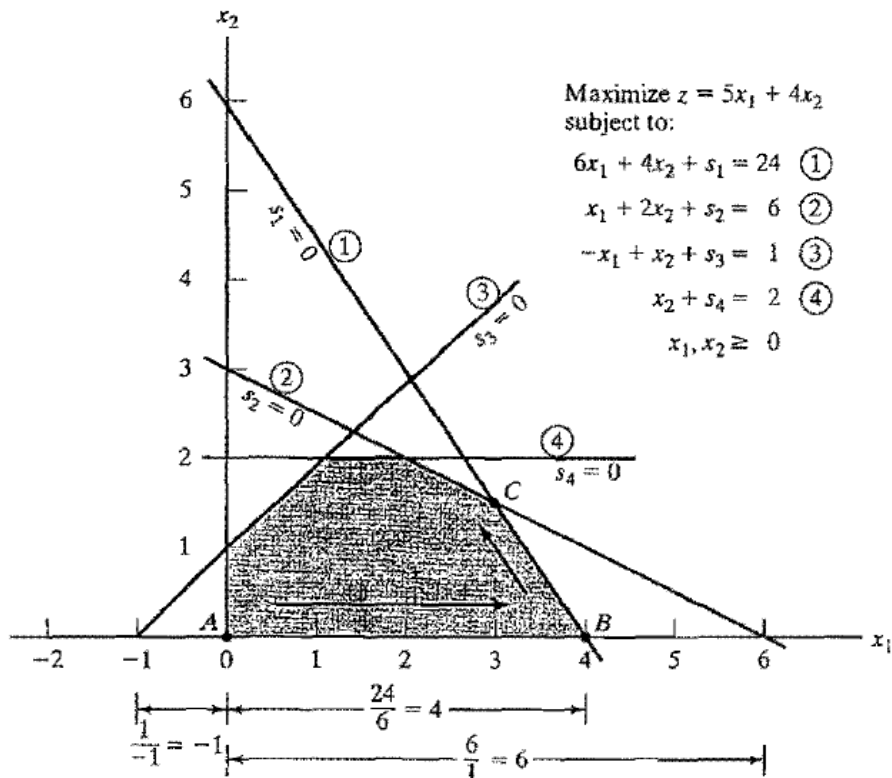
This information is shown in the tableau by listing the basic variables in the **leftmost Basic** column and their values in the **rightmost Solution** column. In effect, the tableau defines the current corner point by specifying its basic variables and their values, as well as the corresponding value of the objective function, z . Remember that **the nonbasic variables** (those not listed in the *Basic* column) always equal **zero**.

Is the starting solution optimal? The objective function $z = 5x_1 + 4x_2$ shows that the solution

can be improved by increasing X_1 or X_2 , X_1 with the *most positive* coefficient is selected as the *entering variable*. Equivalently, because the simplex tableau expresses the objective function as $z - 5X_1 - 4x_2 = 0$, the entering variable will correspond to the variable with the *most negative* coefficient in the objective equation. This rule is referred to as the **optimality condition**.

The mechanics of determining the leaving variable from the simplex tableau calls for computing the *nonnegative ratios* of the right-hand side of the equations (*Solution* column) to the corresponding constraint coefficients under the entering variable, X_1 as the following table shows. Basic

Basic	Entering X_1	Solution	Ratio (or intercept)	
S1	6	24	$X_1=24/6=4$	Minimum
S2	1	6	$X_1=6/1=6$	
S3	-1	1	$X_1=1/-1=-1$	Ignore
S4	0	2	$X_1=2/0=\infty$	ignore
Conclusion: x_1 enters and s_1 leaves				



The *minimum nonnegative* ratio automatically identifies the current basic variable S_1 as the leaving variable and assigns the entering variable X_1 the new value of 4.

The new solution point B is determined by "swapping" the entering variable X_t and the leaving variable in the simplex tableau to produce the following sets of nonbasic and basic variables:

nonbasic (zero) variables at $B:(s_1, x_2)$
 basic variables at $B:(x_1, s_2, s_3, s_4)$

The swapping process is based on the Gauss-Jordan row operations. It identifies the entering variable column as the pivot **column** and the leaving variable row as the pivot row. The intersection of the pivot column and the pivot row is called the pivot element. The following tableau is a restatement of the starting tableau with its pivot row and column highlighted.

		Inter									
		Basic	Z	X1	X2	S1	S2	S3	S4	Solution	
	z		1	-5	-4	0	0	0	0	0	z-row
leave	S1		0	6	4	1	0	0	0	26	S1-row Pivot row
	S2		0	1	2	0	1	0	0	6	S2-row
	S3		0	-1	1	0	0	1	0	1	S3-row
	S4		0	0	1	0	0	0	1	2	
		Pivot Column									

1. Pivot row

- a. Replace the leaving variable in the *Basic* column with the entering variable.
- b. New pivot row = Current pivot row + Pivot element

2. All other rows, including z

New Row = (Current row) - (Its pivot column coefficient) x (New pivot row)

These computations are applied to the preceding tableau in the following manner:

1. Replace S1 in the *Basic* column with X1:

$$\begin{aligned} \text{New } x_1\text{-row} &= \text{Current } s_1\text{-row} \div 6 \\ &= (1/6)(0 \ 6 \ 4 \ 1 \ 0 \ 0 \ 0 \ 24) \\ &= (0 \ 1 \ 2/3 \ 1/6 \ 0 \ 0 \ 0 \ 4) \end{aligned}$$

2. new z-row = current z-row - (-5) * new x1-row

$$\begin{aligned} &= (1 \ -5 \ -4 \ 0 \ 0 \ 0 \ 0 \ 0) - (-5) * (0 \ 1 \ 2/3 \ 1/6 \ 0 \ 0 \ 0 \ 4) \\ &= (1 \ 0 \ -2/3 \ 5/6 \ 0 \ 0 \ 0 \ 20) \end{aligned}$$

3. new s2-row = current s2-row - (1) * new x1-row

$$\begin{aligned} &= (0 \ 1 \ 2 \ 0 \ 1 \ 0 \ 0 \ 6) - (1) * (0 \ 1 \ 2/3 \ 1/6 \ 0 \ 0 \ 0 \ 4) \\ &= (0 \ 0 \ 4/3 \ -1/6 \ 1 \ 0 \ 0 \ 2) \end{aligned}$$

4. new s3-row = current s4-row - (-1) * new x1-row

$$= (0 \ -1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1) - (-1) * (0 \ 1 \ 2/3 \ 1/6 \ 0 \ 0 \ 0 \ 4)$$

$$=(0 \ 0 \ 5/3 \ 1/6 \ 0 \ 1 \ 0 \ 5)$$

$$\begin{aligned} 5\text{- new } s4\text{-row} &= \text{current } s4\text{-row} - (0) * \text{new } x1\text{-row} \\ &= (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 2) - (0)(0 \ 1 \ 2/3 \ 1/6 \ 0 \ 0 \ 0 \ 4) \\ &= (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 2) \end{aligned}$$

The new basic solution is $(X1 > .)2, S3, S4$. and the new tableau becomes

inter										
	Basic	Z	X1	X2	S1	S2	S3	S4	Solution	
	z	1	0	-2/3	5/6	0	0	0	20	z-row
	x1	0	1	2/3	1/6	0	0	0	4	x1-row
leave	S2	0	0	4/3	-1/6	1	0	0	2	S2-row Pivot row
	S3	0	0	5/3	1/6	0	1	0	5	S3-row
	S4	0	0	1	0	0	0	1	2	S4-row
Pivot Column										

Observe that the new tableau has the same properties as the starting tableau. When we set the new nonbasic variables $X2$ and $S1$ to zero, the *Solution* column automatically yields the new basic solution $(x1=4, s2=2, s3=5, s4=2)$. This "conditioning" of the tableau is the result of the application of the Gauss-Jordan row operations. The corresponding new objective value is $z = 20$, which is consistent with

$$\begin{aligned} \text{new } z &= \text{old } z + \text{new } x1\text{-value} * \text{its objective coefficient} \\ &= 0 + 4 * 5 = 20 \end{aligned}$$

In the last tableau, the *optimality condition* shows that $x2$ is the entering variable. The feasibility condition produces the following

Basic	Entering X2	Solution	Ratio (or intercept)	
x1	2/3	4	$X2=4 \div (2/3)=6$	
S2	4/3	2	$X2=2 \div (4/3)=1.5$	Minimum
S3	5/3	5	$X2=5 \div (5/3)=3$	
S4	1	2	$X2=2 \div 1=2$	
Conclusion: $x2$ enters and $s2$ leaves				

Replacing $S2$ in the *Basic* column with entering $X2$, the following Gauss-Jordan row operations are applied

1. New pivot $x2\text{-rOW} = \text{Current } S2\text{-row} \div (4/3)$
2. New $z\text{-row} = \text{Current } z\text{-row} - (-2/3) \times \text{New } x2\text{-row}$

3. New x1-row = Current X1-row - (2/3) x New x2-row
4. New syrow = Current s3- row – (5/3) x New x2-row
5. New S4-row = Current S4-row (1) X New x2-row

Basic	Z	X1	X2	S1	S2	S3	S4	Solution	
z	1	0	0	3/4	1/2	0	0	21	z-row
x1	0	1	0	1/4	-1/2	0	0	3	x1-row
x2	0	0	1	-1/8	3/4	0	0	3/2	x2-row
S3	0	0	0	3/8	-5/4	1	0	5/2	S3-row
S4	0	0	0	1/8	-3/4	0	1	1/2	S4-row

Based on the optimality condition, *none* of the z-row coefficients associated with the nonbasic variables, S1 and S2, are negative. Hence, the last tableau is optimal.

The solution also gives the status of the resources. A resource is designated as **scarce** if the activities (variables) of the model use the resource completely. Otherwise, the resource is **abundant**. This information is secured from the optimum tableau by checking the value of the slack variable associated with the constraint representing the resource. If the slack value is zero, the resource is used completely and, hence, is classified as **scarce**. Otherwise, a positive slack indicates that the resource is **abundant**.

The following table classifies the constraints of the model:

Resource	Slack value	status
Raw material, M1	S1=0	Scarce
Raw material, M2	S2=0	Scarce
Market limit	S3=5/2	Abundant
Demand limit	S4=1/2	abundant

3.Summary of the Simplex Method

In minimization problems, the *optimality condition* calls for selecting the entering variable as the nonbasic variable with the most *positive* objective coefficient in the objective equation, the exact opposite rule of the maximization case. This follows because $\max z$ is equivalent to $\min (-z)$. As for the *feasibility condition* for selecting the leaving variable, the rule remains unchanged.

Optimality condition. The entering variable in a maximization (minimization) problem is the *nonbasic* variable having the most negative (positive) coefficient in the z-row. Ties are broken arbitrarily. The optimum is reached at the iteration where all the z-row coefficients of the nonbasic variables are nonnegative (nonpositive).

Feasibility condition. For both the maximization and the minimization problems,

the leaving variable is the *basic* variable associated with the smallest nonnegative ratio (with *strictly positive* denominator). Ties are broken arbitrarily.

Gauss-Jordan row operations.

1. Pivot row
 - a. Replace the leaving variable in the *Basic* column with the entering variable.
 - b. New pivot row = Current pivot row + Pivot element
2. All other rows, including *z*

$$\text{New row} = (\text{Current row}) - (\text{pivot column coefficient}) \times (\text{New pivot row})$$

The **steps** of the simplex method are

Step 1: Determine a starting basic feasible solution.

Step 2: Select an *entering variable* using the optimality condition. Stop if there is no entering variable; the last solution is optimal. Else, go to step 3.

Step 3: Select a *leaving variable* using the feasibility condition.

Step 4: Determine the new basic solution by using the appropriate Gauss-Jordan computations. Go to step 2.

Example:

$$\text{Minimize } z = -5x_1 - 3x_2 + 2x_3$$

Subject to :

$$X_1 + x_2 + x_3 \leq 5$$

$$X_1 - 2x_2 - 2x_3 \leq 4$$

$$3x_1 + 3x_2 + 2x_3 \leq 15$$

$$X_1, x_2, x_3 \geq 0$$

Solving these equation and found x_1, x_2 and x_3 values?

Solution:

Step one: convert inequality to equality equation.

$$\text{Minimize } z = 5x_1 + 3x_2 - 2x_3$$

Subject to :

$$X_1 + x_2 + x_3 + s_1 = 5$$

$$X_1 - 2x_2 - 2x_3 + s_2 = 4$$

$$3x_1 + 3x_2 + 2x_3 + s_3 = 15$$

$$X_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

step two: select pivot column which is represent max. positive value in object function.

Here column x_1 is inter variable represent.

		inter							
Basic variable	Z	X1	X2	X3	S1	S2	S3	solution	
Z	1	5	3	-2	0	0	0	0	z-row
S1	0	1	1	1	1	0	0	5	S1-row
S2	0	1	-2	-2	0	1	0	4	S2-row
S3	0	3	3	2	0	0	1	15	S3-row
Pivot column									

To select pivot row which represent min. ratio of solution value divide by coefficient of pivot column as following and s1 is leave variable.

Basic	Entering X1	Solution	Ratio (or intercept)	
S1	1	5	$X1=5/1=5$	
S2	1	4	$X1=4/1=4$	minimize
S3	3	15	$X1=15/3=5$	
Conclusion: x1 enters and s2 leaves				

Both pivot column and pivot row are represent as:

			inter							
	Basic variable	Z	X1	X2	X3	S1	S2	S3	solution	
	Z	1	5	3	-2	0	0	0	0	
	S1	0	1	1	1	1	0	0	5	
leave	X1	0	1	-2	-2	0	1	0	4	Pivot row
	S3	0	3	3	2	0	0	1	15	
Pivot column										

These computations are applied to the preceding tableau in the following manner:

1. Replace S2 in the *Basic* column with X1:

$$\begin{aligned} \text{New x1-row} &= \text{Current s1-row} \div 1 \\ &= (1/1)(0 \ 1 \ -2 \ -2 \ 0 \ 1 \ 0 \ 4) \\ &= (0 \ 1 \ -2 \ -2 \ 0 \ 1 \ 0 \ 4) \end{aligned}$$

2. new z-row=current z-row-(5)*new x1-row

$$\begin{aligned} &= (1 \ 5 \ 3 \ -2 \ 0 \ 0 \ 0 \ 0) - (5) * (0 \ 1 \ -2 \ -2 \ 0 \ 1 \ 0 \ 4) \\ &= (1 \ 0 \ 13 \ 8 \ 0 \ -5 \ 0 \ -20) \end{aligned}$$

3. new s1-row =current s1-row -(1)*new x1-row

$$\begin{aligned} &= (0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 5) - (1) * (0 \ 1 \ -2 \ -2 \ 0 \ 1 \ 0 \ 4) \\ &= (0 \ 0 \ 3 \ 3 \ 1 \ -1 \ 0 \ 1) \end{aligned}$$

4. new s3-row=current s3-row -(3)*new x1-row

$$\begin{aligned} &= (0 \ 3 \ 3 \ 2 \ 0 \ 0 \ 1 \ 15) - (3) * (0 \ 1 \ -2 \ -2 \ 0 \ 1 \ 0 \ 4) \\ &= (0 \ 0 \ 9 \ 8 \ 0 \ -3 \ 1 \ 3) \end{aligned}$$

select pivot column which is represent max. positive value in object function.

Here column x2 is inter variable represent.

			inter							
Basic variable	Z	X1	X2	X3	S1	S2	S3	solution		
Z	1	0	13	8	0	-5	0	-20		z-row
S1	0	0	3	3	1	-1	0	1		S1-row
X1	0	1	-2	-2	0	1	0	4		X1-row
S3	0	0	9	8	0	-3	1	3		S3-row
Pivot column										

To select pivot row which represent min. ratio of solution value divide by coefficient of pivot column as following and s1 is leave variable

Basic	Entering X2	Solution	Ratio (or intercept)	
S1	3	1	$X1=1/3=1/3$	minimize
X2	-2	4	$X1=4/-2=-2$	ignore
S3	9	3	$X1=15/3=3/9=1/3$	
Conclusion: x2 enters and s1 leaves				

Note: both s1 and s3 can choose, we select s1

	Basic variable	Z	inter X1	inter X2	X3	S1	S2	S3	solution	
	Z	1	0	13	8	0	-5	0	-20	
leave	X2	0	0	3	3	1	-1	0	1	Pivot row
	X1	0	1	-2	-2	0	1	0	4	
	S3	0	0	9	8	0	-3	1	3	
Pivot column										

These computations are applied to the preceding tableau in the following manner:

1. Replace S1 in the *Basic* column with X2:

$$\begin{aligned} \text{New x2-row} &= \text{Current s1-row} \div 3 \\ &= (1/3)(0 \ 0 \ 3 \ 3 \ 1 \ -1 \ 0 \ 1) \\ &= (0 \ 0 \ 1 \ 1 \ 1/3 \ -1/3 \ 0 \ 1/3) \end{aligned}$$

2. new z-row=current z-row-(13)*new x2-row

$$\begin{aligned} &= (1 \ 0 \ 13 \ 8 \ 0 \ -5 \ 0 \ -20) - (13) * (0 \ 0 \ 1 \ 1 \ 1/3 \ -1/3 \ 0 \ 1/3) \\ &= (1 \ 0 \ 0 \ -5 \ -13/3 \ -5+13/3 \ 0 \ -20-13/3) = (1 \ 0 \ 0 \ -5 \ -13/3 \ -2/3 \ 0 \ -73/3) \end{aligned}$$

3. new x1-row =current x1-row -(-2)*new x2-row

$$\begin{aligned} &= (0 \ 1 \ -2 \ -2 \ 0 \ 1 \ 0 \ 4) - (-2) * (0 \ 0 \ 1 \ 1 \ 1/3 \ -1/3 \ 0 \ 1/3) \\ &= (0 \ 1 \ 0 \ 0 \ 2/3 \ 1-2/3 \ 0 \ 4+2/3) = (0 \ 1 \ 0 \ 0 \ 2/3 \ 1/3 \ 0 \ 14/3) \end{aligned}$$

4. new s3-row=current s3-row -(9)*new x2- row

$$\begin{aligned} &= (0 \ 0 \ 9 \ 8 \ 0 \ -3 \ 1 \ 3) - (9) * (0 \ 0 \ 1 \ 1 \ 1/3 \ -1/3 \ 0 \ 1/3) \\ &= (0 \ 0 \ 0 \ -1 \ -3 \ 0 \ 1 \ 0) \end{aligned}$$

Basic variable	Z	X1	X2	X3	S1	S2	S3	solution	
Z	1	0	0	-5	-13/3	-2/3	0	-73/3	z-row
X2	0	0	1	1	1/3	-1/3	0	1/3	X2-row
X1	0	1	0	0	2/3	1/3	0	14/3	X1-row
S3	0	0	0	-1	-3	0	1		S3-row

$$Z = -73/3$$

$$X1 = 14/3$$

$$X2 = 1/3$$

$$X3 = 0$$