Unbounded Solution

The unbounded solution is explained in the following Example.

Example

Consider the following linear programming problem.

$$\text{Maximize} \quad 5x_1 + 4x_2$$

Subject to:

$$x_1 - x_2 \le 8$$

$$x_1 \le 7$$

$$x_1, x_2 \ge 0.$$

Solution :

Introduce the slack variables s_3 and s_4, so that the inequalities becomes as equation as follows:

$$x_1 + s_3 = 7$$

$$x_1 - x_2 + s_4 = 8$$

$$x_1, x_2, s_3, s_4 \ge 0.$$

The calculation of simplex procedures and tables are as follows:

CB	Basic variables	C_j XB	5 x_1	4 x_2	0 s_3	0 s_4
0	s_3	7	1	0	1	0
0	s_4	8	1	-1	0	1
		z_j-c_j	-5	-4	0	0

CB	Basic variables	C_j XB	5 x_1	4 x_2	0 s_3	0 s_4
5	x_1	7	1	0	1	0
0	s_4	1	0	-1	-1	1
		z_j-c_j	0	-4	5	0

Note that z_2 - c_2 < 0 which indicates x_2 should be introduced as a basic variable in the next iteration. However, both $y_{12} \leq 0$, $y_{22} \leq 0$.

Thus, it is not possible to proceed with the simplex method of calculation any further as we cannot decide which variable will be non-basic at the next iteration. This is the criterion for unbounded solution.

NOTE: If in the course of simplex computation z_j-c_j < 0 but $y_{ij} \leq 0$ for all i then the problem has no finite solution.

But in this case we may observe that the variable x_2 is unconstrained and can be increased arbitrarily. This is why the solution is unbounded.

The solutions of the exercises:

Exercise 1: **Maximize** $f(x,y) = 5x + y$

Subject to

$$x + 2y + z \leq 6$$
$$4x + 3y \leq 120$$
$$x, y \geq 0$$

Solution:

The slack equations are:

$$x + 2y + s_1 = 6$$

$$4x + 3y + s_2 = 120$$

Rewrite the objective function:

$$P = 5x + y \implies -5x - y + P = 0$$

The initial simplex tableau:

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \\ \hline 1 & 2 & 1 & 0 & 0 & 6 \\ 4 & 3 & 0 & 1 & 0 & 120 \\ \hline -5 & -1 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator in the last row is - **5** and so the pivot column is the **x**-column:

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \\ \hline 1 & 2 & 1 & 0 & 0 & 6 \\ 4 & 3 & 0 & 1 & 0 & 120 \\ \hline -5 & -1 & 0 & 0 & 1 & 0 \\ \uparrow & & & & & \end{array}$$

$$p.c.: \quad 6/1 = 6,\ 120/4 = 30$$

The smallest non-negative quotient is **6**. Pivot about **1** in row **1**:

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \\ \hline 1 & 2 & 1 & 0 & 0 & 6 \\ 0 & -5 & -4 & 1 & 0 & 96 \\ \hline 0 & 9 & 5 & 0 & 1 & 30 \end{array}$$

No negative indicators exist and so the maximum has been reached. The solution is:

$$x = 6,\ y = 0,\ s_1 = 0,\ s_2 = 96,\ P = 30$$

Exercise 2:

Maximize: $P = .05x + .09y + .08z$

subject to the following constraints:

$$x + y + z \le 150$$
$$x \le 75$$
$$y \le 75$$
$$z \le 75$$

The slack equations are:

$$\begin{aligned} x + y + z + s_1 &= 150 \\ x + s_2 &= 75 \\ y + s_3 &= 75 \\ z + s_4 &= 75 \\ -.05x - .09y - .08z + P &= 0 \end{aligned}$$

The initial tableau is:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 150 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & -.09 & -.08 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The smallest non-negative quotient is 75 and so pivot about 1 in row 3:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 150 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & -.09 & -.08 & 0 & 0 & 0 & 0 & 1 & 0 \\ & \uparrow & & & & & & & \\ & p.c. & & & & & & & \end{array}$$

$150/1,\ 75/1$

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ 1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 75 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & 0 & -.08 & 0 & 0 & .09 & 0 & 1 & 6.75 \end{array}$$

The smallest non-negative quotient is 75 and occurs in rows 1 and 4. Arbitrarily take 1 in row 4 pivoting purposes:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ 1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 75 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & 0 & -.08 & 0 & 0 & .09 & 0 & 1 & 6.75 \\ & & \uparrow & & & & & & \\ & & p.c. & & & & & & \end{array}$$

$75/1$

The resulting tableau is:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & 0 & 0 & 0 & 0 & .09 & .08 & 1 & 12.75 \end{array}$$

The smallest non-negative quotient is **0** and occurs in row 1:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & 0 & 0 & 0 & 0 & .09 & .08 & 1 & 12.75 \\ \uparrow & & & & & & & & \\ p.c. & & & & & & & & \end{array}$$

$0/1,\ 75/1$

Pivoting yields the following tableau. Since there are no negative entries in the bottom row, this tableau gives the solution.

x	y	z	s_1	s_2	s_3	s_4	P	
1	0	0	1	0	−1	−1	0	0
0	0	0	−1	1	1	1	0	75
0	1	0	0	0	1	0	0	75
0	0	1	0	0	0	1	0	75
0	0	0	.05	0	.04	.03	1	16.50

Max. income is ₨.16,500 **when** $x = 0$ **is in stocks,** $y =$ ₨.75000 **is in bonds, and** $z = \$75{,}000$ **is in a money market**

Exercise 3:

Maximize $f = 5x + 10y$ subject to $\begin{cases} x + y \le 20 \\ 2x - y \ge 10 \\ x, y \ge 0 \end{cases}$

Change the second constraint to a $\le$

$$\begin{cases} x + y \le 20 \\ -2x + y \le -10 \\ x, y \ge 0 \end{cases}$$

Setup the initial simplex tableau

$$\begin{array}{cccccc} x & y & s_1 & s_2 & f & \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ -2 & 1 & 0 & 1 & 0 & -10 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Eliminate the negative entry in the right column:

Look for the "most" negative entry to the left of the right column. $\therefore$ the first column is the pivot column. Since there is a pos. ratio, we pivot about the 1 in row 1.

$$\begin{array}{cccccc} x & y & s_1 & s_2 & f & \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 3 & 2 & 1 & 0 & 30 \\ 0 & -5 & 5 & 0 & 1 & 100 \end{bmatrix}$$

The tableau is in standard form. Pivot in the second column and pivot about the 3 in row 2.

$$\begin{array}{cccccc} x & y & s_1 & s_2 & f & \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 1 & 2/3 & 1/3 & 0 & 10 \\ 0 & -5 & 5 & 0 & 1 & 100 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1/3 & -1/3 & 0 & 10 \\ 0 & 1 & 2/3 & 1/3 & 0 & 10 \\ 0 & 0 & 25/3 & 5/3 & 1 & 150 \end{bmatrix}$$

$$\begin{matrix} x & y & s_1 & s_2 & f & \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 1 & 2/3 & 1/3 & 0 & 10 \\ 0 & -5 & 5 & 0 & 1 & 100 \end{bmatrix}$$

$$\begin{matrix} x & y & s_1 & s_2 & f & \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1/3 & -1/3 & 0 & 10 \\ 0 & 1 & 2/3 & 1/3 & 0 & 10 \\ 0 & 0 & 25/3 & 5/3 & 1 & 150 \end{bmatrix}$$

The maximum value is 150 when $x = 10,\ y = 10$