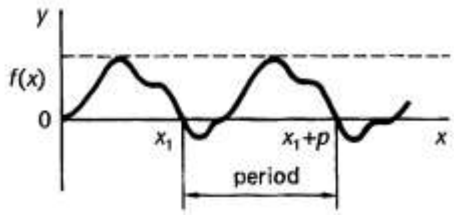
Fourier Series

We have seen earlier that many functions can be expressed in the form of infinite series. Problems involving various forms of oscillations are common in fields of modern technology and *Fourier series*, with which we shall now be concerned, enable us to represent a periodic function as an infinite trigonometrical series in sine and cosine terms. One important advantage of a Fourier series is that it can represent a function containing discontinuities, whereas Maclaurin's and Taylor's series require the function to be continuous throughout.

Periodic functions

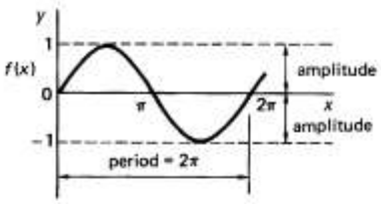
A function $f(x)$ is said to be *periodic* if its function values repeat at regular intervals of the independent variable. The regular interval between repetitions is the *period* of the oscillations.



Graphs of $y = A\sin nx$

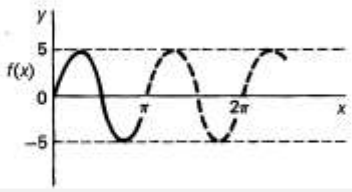
(a) $y = \sin x$

The obvious example of a periodic function is $y = \sin x$, which goes through its complete range of values while x increases from 0° to 360°. The period is therefore 360° or 2π radians and the amplitude, the maximum displacement from the position of rest, is 1.



(b) $y = 5\sin 2x$

The amplitude is 5.
The period is 180° and there are thus 2 complete cycles in 360°.



(c) $y = A \sin nx$

Thinking along the same lines, the function $y = A \sin nx$ has amplitude ....; period ....; and will have .... complete cycles in 360°.

$$\text{amplitude} = A;\ \text{period} = \frac{360°}{n} = \frac{2\pi}{n};\ n \text{ cycles in } 360°$$

Graphs of $y = A \cos nx$ have the same characteristics.

By way of revising earlier work, then, complete the following short exercise.

Exercise

In each of the following, state (a) the amplitude and (b) the period.

1. $y = 3 \sin 5x$
2. $y = 2 \cos 3x$
3. $y = \sin \frac{x}{2}$
4. $y = 4 \sin 2x$
5. $y = 5 \cos 4x$
6. $y = 2 \sin x$
7. $y = 3 \cos 6x$
8. $y = 6 \sin \frac{2x}{3}$

Harmonics

A function $f(x)$ is sometimes expressed as a series of a number of different sine components. The component with the largest period is the *first harmonic*, or *fundamental* of $f(x)$.

$y = A_1 \sin x$ is the first harmonic or fundamental

$y = A_2 \sin 2x$ is the second harmonic

$y = A_3 \sin 3x$ is the third harmonic, etc.

and in general

$y = A_n \sin nx$ is the .... harmonic, with amplitude .... and period ....

$$n\text{th harmonic; amplitude } A_n;\ \text{period} = \frac{2\pi}{n}$$

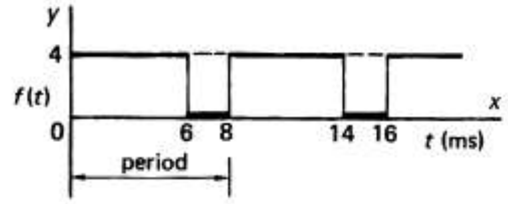
Non-sinusoidal periodic functions

Although we introduced the concept of a periodic function via a sine curve, a function can be periodic without being obviously sinusoidal in appearance.

Example

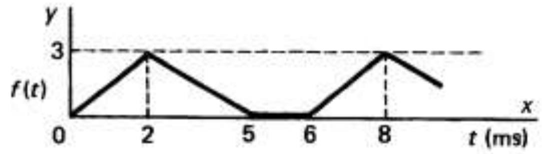
In the following cases, the x-axis carries a scale of t in milliseconds.

(a)



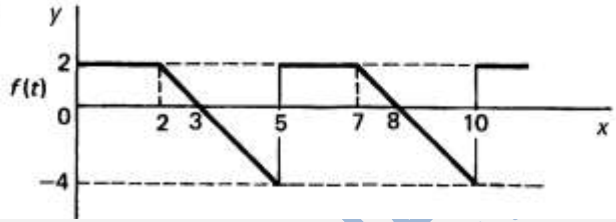
period = 8 ms

(b)



period = ...

(c)



period = ...

Orthogonal functions

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \le x \le b$ and

$$\int_a^b f(x)g(x)\,dx = 0$$

then we say that the two functions are **orthogonal** to each other on the interval $a \le x \le b$. In the previous frames we have seen that the trigonometric functions $\sin nx$ and $\cos nx$ where $n = 0, 1, 2, \ldots$ form an infinite collection of periodic functions that are mutually orthogonal on the interval $-\pi \le x \le \pi$, indeed on any interval of width 2π. That is

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0 \quad \text{for } m \ne n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0 \quad \text{for } m \ne n$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = 0$$

Fourier series and Fourier Coefficient

Let f(x) be defined in the interval (-L,L) and outside of this interval by f(x+2L) =f(x) .i.e., f(x) is 2L-periodic. It is through this avenue that a new function on an infinite set of real numbers is created from the image on(-L,L). The Fourier series or Fourier expansion corresponding to f(x) is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where the Fourier coefficients a_0, a_n and b_n are

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

For n=1,2,3...

Example: Determine the Fourier coefficient a_0,

Integrate both sides of the Fourier series (1), i.e.,

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \frac{a_0}{2}\,dx + \int_{-L}^{L} \sum_{n=1}^{\infty}\left\{a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right\} dx$$

Now $\int_{-L}^{L} \frac{a_0}{2}\,dx = a_0 L$, $\int_{-l}^{L} \sin\frac{n\pi x}{L}\,dx = 0$, $\int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx = 0$, therefore, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

Example

If the series $A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots$,

$$(a)\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad (b)\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad (c)\ A = \frac{a_0}{2}.$$

(a) Multiplying

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

by $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 13.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= a_m L \qquad \text{if } m \neq 0\end{aligned}$$

Thus $$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

Example: let us consider the function

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)dx + \frac{1}{\pi}\int_{0}^{\pi}(1)dx$$

$$= -x|_{-\pi}^{0} + x|_{0}^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)dx$$

$$= \frac{1}{\pi}\left([-\frac{1}{n}\sin(nx)]_{-\pi}^{0} + [\frac{1}{n}\sin(nx)]_{0}^{\pi}\right)$$

$$= \frac{1}{n\pi}(-[\sin(0) - \sin(-\pi)] + [\sin(\pi) - \sin(0)])$$

$$= 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)dx$$

$$= \frac{1}{\pi}\left([\frac{1}{n}\cos(nx)]_{-\pi}^{0} + [-\frac{1}{n}\cos(nx)]_{0}^{\pi}\right)$$

$$= \frac{1}{n\pi}([\cos(0) - \cos(-n\pi)] + [\cos(0) - \cos(n\pi)])$$

$$= \frac{2}{n\pi}(1 - \cos(n\pi)) = \frac{2}{n\pi}(1 - (-1)^n).$$

n	B
1	$\frac{4}{\pi}$
2	0
3	$\frac{4}{3\pi}$
4	0
5	$\frac{4}{5\pi}$

Thus the Fourier series of f is given as

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n)\sin(nx).$$

Example: Let us consider the function f defined as follows

$$f(x) = \begin{cases} 0, & -2 \le x < 0 \\ 2-x, & 0 < x \le 2. \end{cases}$$

By using the formula of a_0, a_n and b_n, we find that

$$a_0 = \frac{1}{4}\int_0^2 (2-x)dx,$$

$$a_k = \frac{1}{2}\int_0^2 (2-x)\cos\frac{k\pi x}{2}dx,\ k = 1, 2, \ldots$$

and

$$b_k = \frac{1}{2}\int_0^2 (2-x)\sin\frac{k\pi x}{2}dx,\ k = 1, 2, \ldots$$

Evaluating these integrals gives

$$a_0 = \frac{1}{2},\ a_k = \frac{2}{k^2\pi^2}[1-(-1)^k] \text{ and } b_k = \frac{2}{k\pi}$$

where use has been made of the fact that $\cos(k\pi) = (-1)^k$ and $\sin(k\pi) = 0$. Thus the Fourier series becomes

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{[1-(-1)^k]}{k^2\pi}\cos\frac{k\pi x}{2} + \frac{1}{k}\sin\frac{k\pi x}{2}\right).$$

Dirichlet Conditions

Suppose that

(1) $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$

(2) $f(x)$ is periodic outside $(-L, L)$ with period $2L$

(3) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

Then the series (*1*) with Fourier coefficients converges to

(*a*) $f(x)$ if x is a point of continuity

(*b*) $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Example:

If the following functions are defined over the interval $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$, state whether or not each function can be represented by a Fourier series.

1 $f(x) = x^3$

2 $f(x) = 4x - 5$

3 $f(x) = \frac{2}{x}$

4 $f(x) = \frac{1}{x-5}$

5 $f(x) = \tan x$

6 $f(x) = y$ where $x^2 + y^2 = 9$

Solution:

1 Yes

2 Yes

3 No: infinite discontinuity at $x = 0$

4 Yes

5 No: infinite discontinuity at $x = \pi/2$

6 No: two valued

Even and Odd Functions "Half-Range Expansions"

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval (0,L) which is half of the interval(-L,L) thus accounting for the name half range] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely,(-L,0). In such case, we have

$a_0=0$

$$a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{for half range sine series}$$

$$b_n = 0, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{for half range cosine series}$$

Fourier Sine series:

An *odd function* is a function with the property $f(-x) = -f(x)$.

For example :

1. $f(x) = x^3$. let $x = -1$, then $(-1)^3 = -(1)^3$
2. $f(x) = \sin(x)$. let $x = -\pi/2$, then $\sin(-\pi/2) = -\sin(\pi/2)$.

Note

1. The integral of an odd function over a symmetric interval is zero.
2. Since $a_n = 0$, all the cosine functions will not appear in the Fourier series of an odd function. The Fourier series of an odd function is an infinite series of Odd functions

Let us calculate the Fourier coefficients of an odd function:

$a_0=an=0$

but $bn \neq 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{for half range sine series}$$

$$\mathbf{f(x) = \sum_{n=1}^{\infty} bn \sin\frac{n\pi x}{L} dx}$$

Fourier Cosine Series:

An even function is a function with the property $f(-x) = f(x)$. The sine coefficients of a Fourier series will be zero for an even function,

$f(x) = x^2$, let $x = -1$, then (-1)2 = (1)2
$f(x) = \cos(x)$, let $x = -\pi$, then $\cos(-\pi) = \cos(\pi)$.
The Fourier series of an even function is an infinite series of even functions (cosines):

Let us calculate the Fourier coefficients of an odd function:

bn =0

But

a_0=an≠ 0

$$a_0 = \frac{2}{L}\int_0^l f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad \text{for half range cosine series}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\ \cos\frac{n\pi x}{L}dx$$

Example: Let us consider the function $f(x) = 1$ on $[0, \pi]$. The Fourier cosine series has coefficients

$$a_0 = \frac{2}{\pi}\int_0^\pi 1dx = 2$$

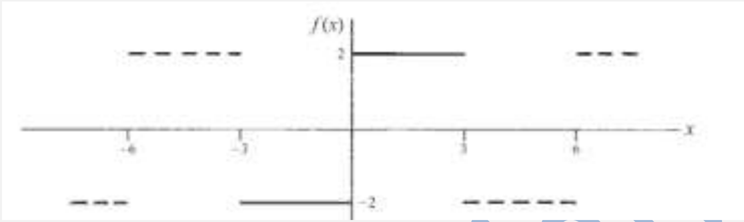
$$a_n = \frac{2}{\pi}\int_0^\pi \cos(nx)dx = 0$$

Then $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = 1.$

Example: Classify each of the following functions according as they are even, odd, or neither even nor odd.

$$(a)\ f(x)=\begin{cases} 2 & 0<x<3 \\ -2 & -3<x<0 \end{cases} \qquad \text{Period}=6$$

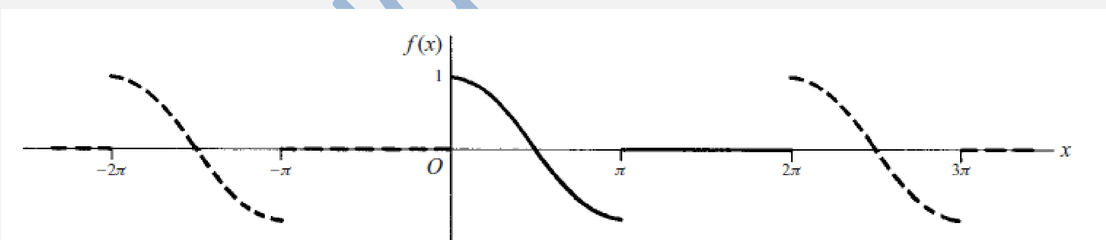
Sol: From fig.1 below it is seen that f(-x)= -f(x), so that the function is odd.



Fig(1)

$$(b)\ f(x)=\begin{cases} \cos x & 0<x<\pi \\ 0 & \pi<x<2\pi \end{cases} \qquad \text{Period}=2\pi$$

Sol: From fig.2 below it is seen that is neither even nor odd.

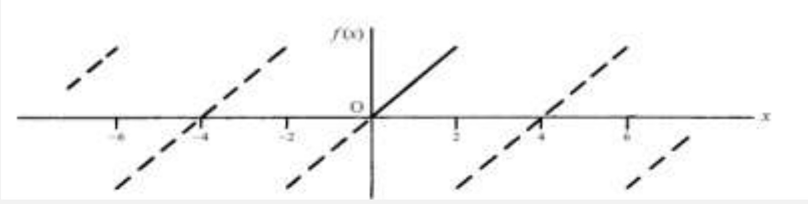


Fig(2)

Example: Expand $f(x)=x,\quad 0<x<2$ **, in a half range**

a) Sine series, (b) cosine series.

(a) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 3, below. This is sometimes called the odd extension of f (x) , Then $2L=4,\ L=2$



Thus $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$
$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then
$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$
$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

b- H.W

Example (H.W)

f (x) = sin (x), 0< x < π

PARSEVAL'S IDENTITY

If an and bn are the Fourier coefficients corresponding to f(x) and if f(x) satisfies the Dirichlet conditions

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Complex Fourier series

The complex exponential of Fourier series is obtained by substitution the exponential equivalent of the **Cosine** and **Sine** into the original form of Series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} \quad \text{and} \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

$$\cos\theta + j\sin\theta = e^{j\theta} \text{and } \cos\theta - j\sin\theta = e^{-j\theta}$$

1/i=-i ; i^2=-1

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{\frac{in\pi t}{L}}+e^{\frac{-in\pi t}{L}}}{2} + b_n \frac{e^{\frac{in\pi t}{L}}-e^{\frac{-in\pi t}{L}}}{2i}\right)$$

If we define

$$C_0 = \frac{a_0}{2};\ C_n = \frac{a_n - ib_n}{2};\ C_{-n} = \frac{a_n + ib_n}{2}$$

The last series can be written

$$f(x) = \sum_{-\infty}^{\infty} \boldsymbol{C_n e^{\frac{in\pi x}{L}}}$$

$$C_0 = \frac{a_0}{2} = \frac{1}{2L}\int f(x)dx$$

$$C_n = \frac{1}{2L}\int f(x)\, \boldsymbol{e^{\frac{-in\pi x}{L}}}$$

$$C_{-n} = \frac{1}{2L}\int f(x)\, \boldsymbol{e^{\frac{in\pi x}{L}}}$$

Example: Find the complex form of Fourier series whose definition in one period

f(t)= e^{-t} -1 < t < 1

Sol:

2L=2 →L=1

$$C_n = \frac{1}{2L}\int f(x)\, e^{\frac{-in\pi x}{L}}$$

$$C_n = \frac{1}{2}\int e^{-t}\, e^{\frac{-in\pi t}{L}}$$

$$= \frac{1}{2}\left(\frac{e^{-(1+in\pi)t}}{-(1+in\pi)}\right)$$

$$\frac{e.e^{in\pi} - e^{-1}.e^{-in\pi}}{-2(1+in\pi)}$$

$$e^{i\pi} = \cos\pi + \mathrm{i}\sin\pi = -1$$

$$e^{in\pi} = (-1)^n$$

$$C_n = \frac{(-1)^n Sinh1}{(1+in\pi)} \times \frac{1-in\pi}{1-in\pi}$$

$$C_n = \frac{1 - in\pi\,(-1)^n Sinh1}{(1+n^2\pi^2)}$$

The expansion of f(t) form can be written as:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1-in\pi\ (-1)^n Sinh1\ e^{in\pi t}}{(1+n^2\pi^2)}$$

The expansion can be converted into real trigonometric form

$$C_n = \frac{a_n - ib_n}{2};\ C_{-n} = \frac{a_n + ib_n}{2}$$

$$a_n = C_n + C_{-n}$$

$$b_n = i\,(C_n - C_{-n})$$

$$a_n = \frac{(-1)^n 2\,Sinh1}{(1+n^2\pi^2)}$$

$$b_n = i\left[\frac{1-in\pi\ (-1)^n Sinh1}{(1+n^2\pi^2)} - \frac{1+in\pi\ (-1)^n Sinh1}{(1+n^2\pi^2)}\right] = \frac{2n\pi\ (-1)^n Sinh1}{1+n^2\pi^2}$$

$$C_0 = \frac{a_0}{2};\ a_0 = 2C_0 = Sinh1$$

$$f(t) = Sinh1 - 2Sinh1\left(\frac{\cos\pi t}{1+\pi^2} - \frac{\cos 2\pi t}{1+4\pi^2} + \cdots\right) - 2\pi Sinh1\left(\frac{\sin\pi t}{1+\pi^2} - \frac{2\sin 2\pi t}{1+4\pi^2}\right) + \cdots$$

H.W.

Q1) Find the complex form of Fourier series of the following functions:

1. f(t)= e^t -1 < t < 1

2. f(t)= $\begin{bmatrix} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{bmatrix}$

Applications of Fourier Series in Circuit Analysis

Effective Values and Power

The effective or rms value of the function

$$f(t) = \tfrac{1}{2}a_0 + a_1\cos\omega t + a_2\cos 2\omega t + \cdots + b_1\sin\omega t + b_2\sin 2\omega t + \cdots$$

$$F_{\text{rms}} = \sqrt{(\tfrac{1}{2}a_0)^2 + \tfrac{1}{2}a_1^2 + \tfrac{1}{2}a_2^2 + \cdots + \tfrac{1}{2}b_1^2 + \tfrac{1}{2}b_2^2 + \cdots} = \sqrt{c_0^2 + \tfrac{1}{2}c_1^2 + \tfrac{1}{2}c_2^2 + \tfrac{1}{2}c_3^2 + \cdots}$$

In general, we may write

$$v = V_0 + \sum V_n \sin(n\omega t + \phi_n) \quad \text{and} \quad i = I_0 + \sum I_n \sin(n\omega t + \psi_n)$$

With corresponding effective values of

$$V_{\text{rms}} = \sqrt{V_0^2 + \tfrac{1}{2}V_1^2 + \tfrac{1}{2}V_2^2 + \cdots} \quad \text{and} \quad I_{\text{rms}} = \sqrt{I_0^2 + \tfrac{1}{2}I_1^2 + \tfrac{1}{2}I_2^2 + \cdots}$$

The average power P follows from integration of the instantaneous power, which is given by the product of v and i:

$$p = vi = \left[V_0 + \sum V_n \sin(n\omega t + \phi_n)\right]\left[I_0 + \sum I_n \sin(n\omega t + \psi_n)\right]$$

Since v and i both have period T. **The average may therefore be calculated over one** period of the voltage wave:

$$P = \frac{1}{T}\int_0^T \left[V_0 + \sum V_n \sin(n\omega t + \phi_n)\right]\left[I_0 + \sum I_n \sin(n\omega t + \psi_n)\right] dt$$

Then the average power is

$$P = V_0 I_0 + \tfrac{1}{2} V_1 I_1 \cos\theta_1 + \tfrac{1}{2} V_2 I_2 \cos\theta_2 + \tfrac{1}{2} V_3 I_3 \cos\theta_3 + \cdots$$

Where $\theta_n = \phi_n - \psi_n$ is the angle on the equivalent impedance of the network at the angular frequency n!

$$P = \tfrac{1}{2} V_1 I_1 \cos\theta_1 = V_{\text{eff}} I_{\text{eff}} \cos\theta$$

$$P = V_0 I_0 = VI$$

$$P = P_0 + P_1 + P_2 + \cdots$$

Example: A series RL circuit in which R =5 Ω and L = 20 mH has an applied voltage as in Fig.1:

$$v = 100 + 50\sin\omega t + 25\sin 3\omega t \text{ (V), with } \omega = 500 \text{ rad/s.}$$

Find the current and the average power. Compute the equivalent impedance of the circuit at each frequency found in the voltage function. Then obtain the respective currents.

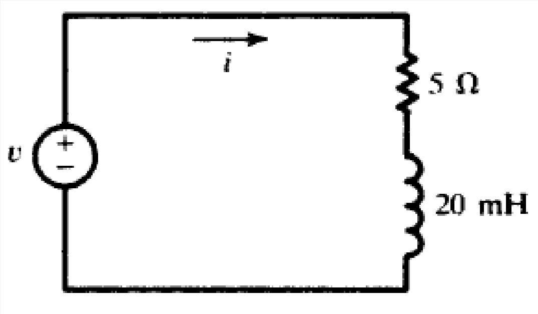


Fig.1

At $\omega = 0$, $Z_0 = R = 5\ \Omega$ and

$$I_0 = \frac{V_0}{R} = \frac{100}{5} = 20\text{ A}$$

At $\omega = 500$ rad/s, $\mathbf{Z}_1 = 5 + j(500)(20 \times 10^{-3}) = 5 + j10 = 11.15\underline{/63.4^\circ}\ \Omega$ and

$$i_1 = \frac{V_{1,\max}}{Z_1}\sin(\omega t - \theta_1) = \frac{50}{11.15}\sin(\omega t - 63.4^\circ) = 4.48\sin(\omega t - 63.4^\circ) \quad \text{(A)}$$

At $3\omega = 1500$ rad/s, $\mathbf{Z}_3 = 5 + j30 = 30.4\underline{/80.54^\circ}\ \Omega$ and

$$i_3 = \frac{V_{3,\max}}{Z_3}\sin(3\omega t - \theta_3) = \frac{25}{30.4}\sin(3\omega t - 80.54^\circ) = 0.823\sin(3\omega t - 80.54^\circ) \quad \text{(A)}$$

The sum of the harmonic currents is the required total response; it is a Fourier series of the type (*8*).

$$i = 20 + 4.48\sin(\omega t - 63.4^\circ) + 0.823\sin(3\omega t - 80.54^\circ) \quad \text{(A)}$$

This current has the effective value

$$I_{\text{eff}} = \sqrt{20^2 + (4.48^2/2) + (0.823^2/2)} = \sqrt{410.6} = 20.25\text{ A}$$

which results in a power in the 5-Ω resistor of

$$P = I_{\text{eff}}^2 R = (410.6)5 = 2053\text{ W}$$

As a check, we compute the total average power by calculating first the power contributed by each harmonic and then adding the results.

At $\omega = 0$: $P_0 = V_0 I_0 = 100(20) = 2000$ W

At $\omega = 500$ rad/s: $P_1 = \frac{1}{2}V_1 I_1 \cos\theta_1 = \frac{1}{2}(50)(4.48)\cos 63.4^\circ = 50.1$ W

At $3\omega = 1500$ rad/s: $P_3 = \frac{1}{2}V_3 I_3 \cos\theta_3 = \frac{1}{2}(25)(0.823)\cos 80.54^\circ = 1.69$ W

Then, $P = 2000 + 50.1 + 1.69 = 2052$ W

Another Method

The Fourier series expression for the voltage across the resistor is

$$v_R = Ri = 100 + 22.4\sin(\omega t - 63.4^\circ) + 4.11\sin(3\omega t - 80.54^\circ) \quad \text{(V)}$$

and

$$V_{Reff} = \sqrt{100^2 + \frac{1}{2}(22.4)^2 + \frac{1}{2}(4.11)^2} = \sqrt{10\,259} = 101.3\ \text{V}$$

Then the power delivered by the source is $P = V_{Reff}^2/R = (10\,259)/5 = 2052$ W.

Example: Find the average power supplied to a network if the applied voltage and resulting current are

$$v = 50 + 50\sin 5\times 10^3 t + 30\sin 10^4 t + 20\sin 2\times 10^4 t \quad \text{(V)}$$

$$i = 11.2\sin(5\times 10^3 t + 63.4^\circ) + 10.6\sin(10^4 t + 45^\circ) + 8.97\sin(2\times 10^4 t + 26.6^\circ) \quad \text{(A)}$$

Sol:

The total average power is the sum of the harmonic powers:

$$P = (50)(0) + \tfrac{1}{2}(50)(11.2)\cos 63.4^\circ + \tfrac{1}{2}(30)(10.6)\cos 45^\circ + \tfrac{1}{2}(20)(8.97)\cos 26.6^\circ = 317.7\ \text{W}$$

Example: Find the trigonometric Fourier series for the half-wave-rectified sine wave shown in Fig. 2 and sketch the line spectrum

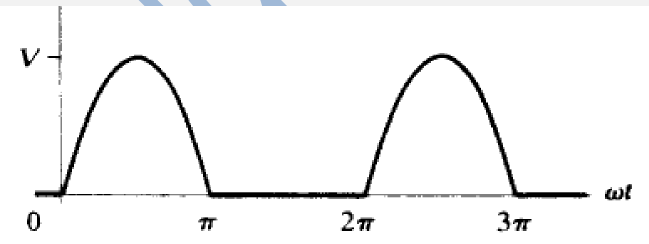


Fig.2

Sol:

The wave shows no symmetry, and we therefore expect the series to contain both sine and cosine terms. Since the average value is not obtainable by inspection, we evaluate a_0 for use in the term a_0=2.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \, d(\omega t) = \frac{V}{\pi}\left[-\cos \omega t\right]_0^{\pi} = \frac{2V}{\pi}$$

Next we determine a_n:

$$a_n = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \cos n\omega t \, d(\omega t)$$

$$= \frac{V}{\pi}\left[\frac{-n \sin \omega t \sin n\omega t - \cos n\omega t \cos \omega t}{-n^2+1}\right]_0^{\pi} = \frac{V}{\pi(1-n^2)}(\cos n\pi + 1)$$

With n even, $a_n = 2V/\pi(1-n^2)$; and with n odd, $a_n = 0$. However, this expression is indeterminate for $n = 1$, and therefore we must integrate separately for a_1.

$$a_1 = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \cos \omega t \, d(\omega t) = \frac{V}{\pi}\int_0^{\pi} \tfrac{1}{2}\sin 2\omega t \, d(\omega t) = 0$$

Now we evaluate b_n:

$$b_n = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \sin n\omega t \, d(\omega t) = \frac{V}{\pi}\left[\frac{n \sin \omega t \cos n\omega t - \sin n\omega t \cos \omega t}{-n^2+1}\right]_0^{\pi} = 0$$

Here again the expression is indeterminate for $n = 1$, and b_1 is evaluated separately.

$$b_1 = \frac{1}{\pi}\int_0^{\pi} V \sin^2 \omega t \, d(\omega t) = \frac{V}{\pi}\left[\frac{\omega t}{2} - \frac{\sin 2\omega t}{4}\right]_0^{\pi} = \frac{V}{2}$$

Then the required Fourier series is

$$f(t) = \frac{V}{\pi}\left(1 + \frac{\pi}{2}\sin \omega t - \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t - \frac{2}{35}\cos 6\omega t - \cdots\right)$$

The spectrum, **Fig. 3**, shows the strong fundamental term in the series and the rapidly decreasing amplitudes of the higher harmonics.

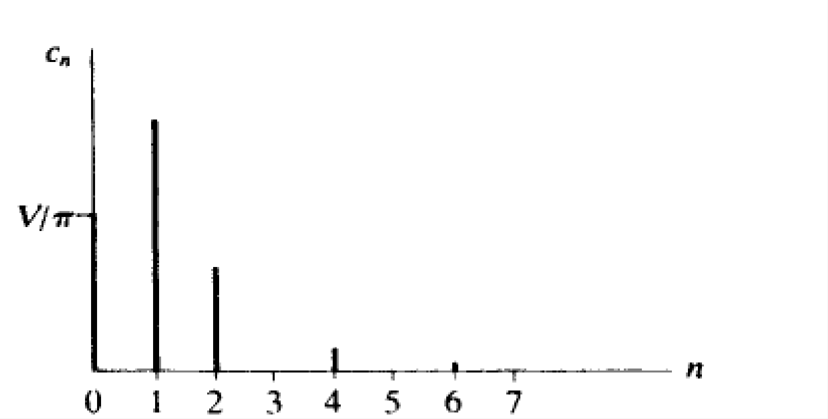


Fig.3

Sheet No 1

1. Prove $\int_{-L}^{L} \sin\frac{k\pi x}{L}\,dx = \int_{-L}^{L} \cos\frac{k\pi x}{L}\,dx = 0 \qquad \text{if } k = 1, 2, 3, \ldots.$

2. (a) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \text{Period} = 10$$

(b) Write the corresponding Fourier series.

(c) How should f(x) be defined at x = -5; x = 0; and x= 5 in order that the Fourier series will Converge to f(x) for **-5 ≤ x ≤ 5?**

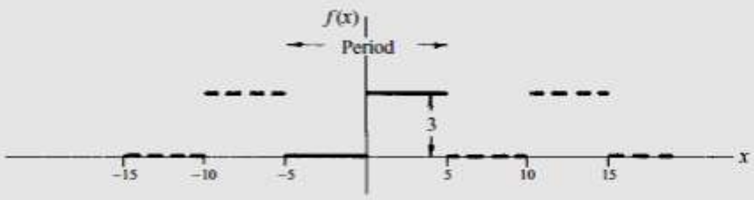


Fig. 13-6

(a) Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c+2L$ as -5 to 5, so that $c = -5$. T

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \qquad \text{if } n \neq 0$$

If $n = 0$, $a_n = a_0 = \frac{3}{5}\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3.$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi}$$

(b) The corresponding Fourier series is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)=\frac{3}{2}+\sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}$$

$$=\frac{3}{2}+\frac{6}{\pi}\left(\sin\frac{\pi x}{5}+\frac{1}{3}\sin\frac{3\pi x}{5}+\frac{1}{5}\sin\frac{5\pi x}{5}+\cdots\right)$$

(c) Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all continuity and to $\dfrac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x=-5$, 0, and 5, which of discontinuity, the series converges to $(3+0)/2=3/2$ as seen from the graph. If we redefi follows,

$$f(x)=\begin{cases}3/2 & x=-5\\ 0 & -5<x<0\\ 3/2 & x=0\\ 3 & 0<x<5\\ 3/2 & x=5\end{cases}\qquad \text{Period}=10$$

then the series will converge to $f(x)$ for $-5\leqq x\leqq 5$.

3.

Expand $f(x)=x^2$, $0<x<2\pi$ in a Fourier series if (a) the period is 2π, (b) the period is not specified.

(a) The graph of f(x) with period 2π is shown in **Fig. 2**

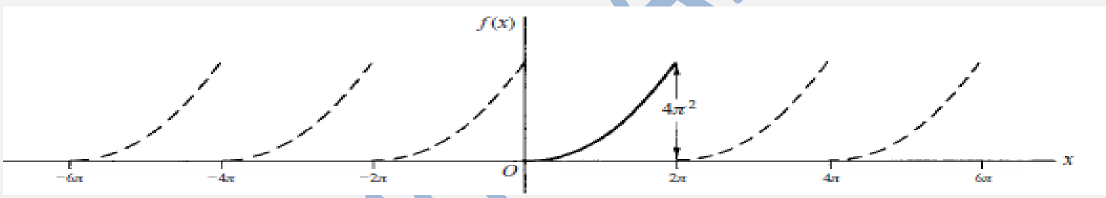


Fig. 2

Period $=2L=2\pi$ and $L=\pi$. Choosing $c=0$, we have

$$a_n=\frac{1}{L}\int_c^{c+2L}f(x)\cos\frac{n\pi x}{L}\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\,dx$$

$$=\frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi}=\frac{4}{n^2},\qquad n\neq 0$$

If $n=0$, $a_0=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}x^2\,dx=\dfrac{8\pi^2}{3}$.

$$b_n=\frac{1}{L}\int_c^{c+2L}f(x)\sin\frac{n\pi x}{L}\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nx\,dx$$

$$=\frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right)-(2x)\left(-\frac{\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi}=\frac{-4\pi}{n}$$

Then $f(x)=x^2=\dfrac{4\pi^2}{3}+\displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right)$.

This is valid for $0<x<2\pi$. At $x=0$ and $x=2\pi$ the series converges to $2\pi^2$.

(b) If the period is not specified, the Fourier series cannot be determined uniquely in general.

4. Expand $f(x)=\sin x,\ 0<x<\pi$, in a Fourier cosine series.

Ans:

$$f(x)=\frac{2}{\pi}-\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1+\cos n\pi)}{n^2-1}\cos nx$$
$$=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1}+\frac{\cos 4x}{4^2-1}+\frac{\cos 6x}{6^2-1}+\cdots\right)$$

5. **For the following graph find the Fourier series**

Ans

$$\frac{4}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right)$$

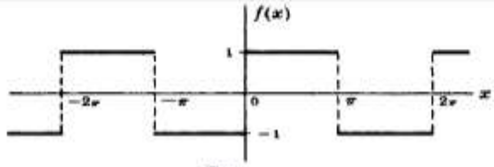


Fig.

Ans

$$\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$$

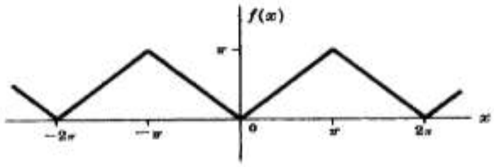
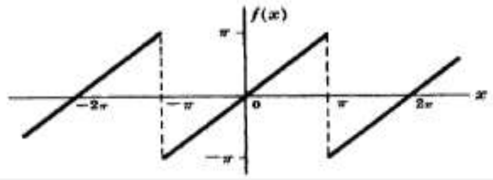


Fig.

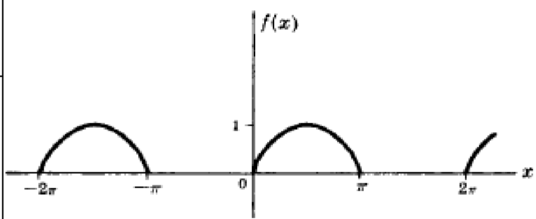
Ans

$$2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\cdots\right)$$



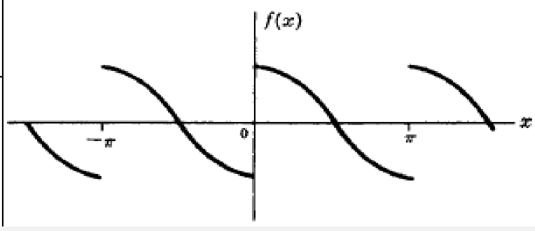
Ans

$$\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\left(\frac{\cos 2x}{1\cdot 3}+\frac{\cos 4x}{3\cdot 5}+\frac{\cos 6x}{5\cdot 7}+\cdots\right)$$

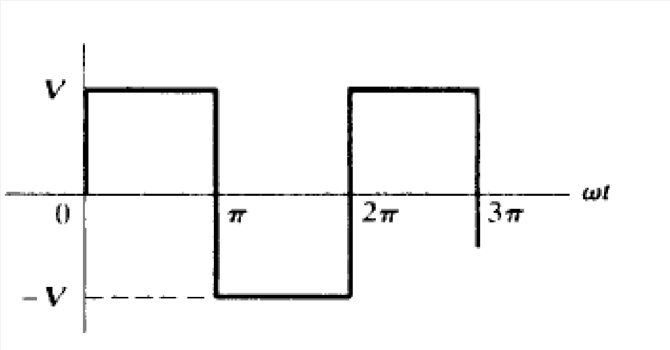


Ans

$$\frac{8}{\pi}\left(\frac{\sin 2x}{1\cdot 3}+\frac{2\sin 4x}{3\cdot 5}+\frac{3\sin 6x}{5\cdot 7}+\cdots\right)$$



6. Find the trigonometric Fourier series for the square wave shown in Fig. below and plot the line spectrum.



Ans:

$$f(t)=\frac{4V}{\pi}\sin\omega t+\frac{4V}{3\pi}\sin 3\omega t+\frac{4V}{5\pi}\sin 5\omega t+\cdots$$

7. Find the exponential Fourier series for the triangular wave shown in Figs. 1 and 2 and sketch the line spectrum.

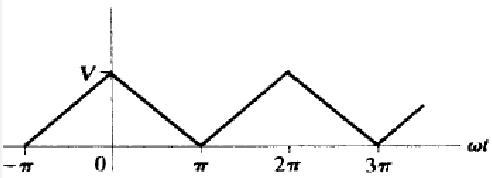


Fig.1

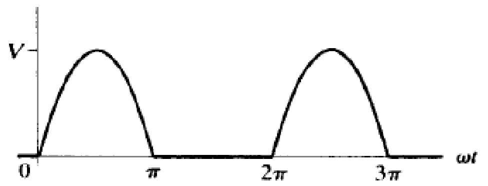


Fig.2

Sol: For Fig.1

In the interval $-\pi<\omega t<0$, $f(t)=V+(V/\pi)\omega t$; and for $0<\omega t<\pi$, $f(t)=V-(V/\pi)\omega t$. The wave is even and therefore the $\mathbf{A}_n$ coefficients will be pure real. By inspection the average value is $V/2$.

$$\mathbf{A}_n=\frac{1}{2\pi}\left\{\int_{-\pi}^{0}[V+(V/\pi)\omega t]e^{-jn\omega t}\,d(\omega t)+\int_{0}^{\pi}[V-(V/\pi)\omega t]e^{-jn\omega t}\,d(\omega t)\right\}$$

$$=\frac{V}{2\pi^2}\left[\int_{-\pi}^{0}\omega t e^{-jn\omega t}\,d(\omega t)+\int_{0}^{\pi}(-\omega t)e^{-jn\omega t}\,d(\omega t)+\int_{-\pi}^{\pi}\pi e^{-jn\omega t}\,d(\omega t)\right]$$

$$=\frac{V}{2\pi^2}\left\{\left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t-1)\right]_{-\pi}^{0}-\left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t-1)\right]_{0}^{\pi}\right\}=\frac{V}{\pi^2n^2}(1-e^{jn\pi})$$

For even n, $e^{jn\pi}=+1$ and $\mathbf{A}_n=0$; for odd n, $\mathbf{A}_n=2V/\pi^2n^2$. Thus the series is

$$f(t)=\cdots+\frac{2V}{(-3\pi)^2}e^{-j3\omega t}+\frac{2V}{(-\pi)^2}e^{-j\omega t}+\frac{V}{2}+\frac{2V}{(\pi)^2}e^{j\omega t}+\frac{2V}{(3\pi)^2}e^{j3\omega t}+\cdots$$

The harmonic amplitudes

$$c_0=\frac{V}{2}\qquad c_n=2|\mathbf{A}_n|=\begin{cases}0 & (n=2,4,6,\ldots)\\ 4V/\pi^2n^2 & (n=1,3,5,\ldots)\end{cases}$$

For **Fig.2**

In the interval $0 < \omega t < \pi$, $f(t) = V \sin \omega t$; and from π to 2π, $f(t) = 0$. Then

$$\mathbf{A}_n = \frac{1}{2\pi}\int_0^{\pi} V \sin \omega t \, e^{-jn\omega t}\, d(\omega t)$$

$$= \frac{V}{2\pi}\left[\frac{e^{-jn\omega t}}{(1-n^2)}(-jn \sin \omega t - \cos \omega t)\right]_0^{\pi} = \frac{V(e^{-jn\pi}+1)}{2\pi(1-n^2)}$$

For even n, $\mathbf{A}_n = V/\pi(1-n^2)$; for odd n, $\mathbf{A}_n = 0$. However, for $n = 1$, the expression for $\mathbf{A}_n$ becomes indeterminate. L'Hôpital's rule may be applied; in other words, the numerator and denominator are separately differentiated with respect to n, after which n is allowed to approach 1, with the result that $\mathbf{A}_1 = -j(V/4)$.

The average value is

$$A_0 = \frac{1}{2\pi}\int_0^{\pi} V \sin \omega t \, d(\omega t) = \frac{V}{2\pi}\Big[-\cos \omega t\Big]_0^{\pi} = \frac{V}{\pi}$$

Then the exponential Fourier series is

$$f(t) = \cdots - \frac{V}{15\pi}e^{-j4\omega t} - \frac{V}{3\pi}e^{-j2\omega t} + j\frac{V}{4}e^{-j\omega t} + \frac{V}{\pi} - j\frac{V}{4}e^{j\omega t} - \frac{V}{3\pi}e^{j2\omega t} - \frac{V}{15\pi}e^{j4\omega t} - \cdots$$

The harmonic amplitudes,

$$c_0 = A_0 = \frac{V}{\pi} \qquad c_n = 2|\mathbf{A}_n| = \begin{cases} 2V/\pi(n^2-1) & (n = 2, 4, 6, \ldots) \\ V/2 & (n = 1) \\ 0 & (n = 3, 5, 7, \ldots) \end{cases}$$